LOwer Secondary School Teacher Training IN MATHematics

LOSSTT-IN-MATH Project

Lower Secondary School Teacher Training in Mathematics

Comparison and Best Practices

Edited by

Franco Favilli

PLUS edizioni

Socrates

Education and Culture

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EC – DG EAC – SOCRATES Program – COMENIUS 2.1 Action

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INTRODUCTION

LOSSTT IN MATH – Lower Secondary School Teacher Training In Mathematics is a project partly supported by the European Commission under the Socrates programme – Comenius 2.1 action (ref. No. 112318-CP-1-2003-1-IT-COMENIUS-C21). It is therefore a project that, by the training of school education staff, aims at enhancing the quality of teaching and strengthening its European dimension.

Rationale of and background to the project

Scientific and technological development is fundamental for competitive knowledge society. This was clearly recognized by the Education Council of the EU which on May 2001 decided to put as one of the top objectives *Increasing recruitment to scientific and technological studies* and by the European Commission that set up a Working Group on *Increasing Participation in Math, Sciences and Technology* – WG on MST. The key issues addressed by the experts in the Working Group included[1]:

- increasing the interest in MST from an early age;
- motivating more people to choose studies and careers in the fields of MST;
- securing a sufficient number of qualified teachers in MST subjects.

There is a general consensus that mathematical knowledge is the foundation of any scientific and technological understanding. Furthermore, in some sense it is a discriminating factor in our developed and globalised society. Great attention must therefore be drawn to the initial training of mathematics teachers.

Generally, school mathematics curricula are almost homogeneous in the European educational systems. However, mathematics teacher training systems in Europe do not reflect this homogeneity. Teacher training systems in the partner countries differ considerably.

At the preparatory stage of the project, preliminary comparison of the educational and teacher training systems as well as investigation of the past and current projects in this area was carried out.

Aims and objectives

In spite of differences in European mathematics teacher training systems the project aims at contributing to greater sharing of good practices in this field. In order to fulfil this task, changes in the curricula for lower secondary school mathematics teacher training are proposed as a result of the piloting of a number of educational modules.

The aim is to influence in a positive way not only teacher training, but also the school reality, through the development of mathematical education projects which intend to be more learner-friendly and attractive to pupils.

Furthermore, the project aims at facilitating the improvement of pupils' achievement.

[1] European Commission, DG for Education and Culture (2004). *Implementation of "Education & Training 2010" work programme – WG on MST Progress Report*. [http://ec.europa.eu/education/policies/2010/doc/math2004.pdf]

Final aim is to move policy makers in education to create the conditions for new mathematics teachers to find it less difficult to get a position in a non-home country, due to a possible common (at least) partly shared curriculum in teacher training institutions throughout the Europe.

Innovative features in the project

The project offers examples of didactical approaches and materials to be used in shaping shared curricula in completely different educational and cultural contexts, with respect to lower secondary school mathematics teacher training.

The proposed use of a foreign language in mathematics classes – hardly experienced in most of the partner countries – should offer the opportunity for higher awareness of the global knowledge required by European citizens.

Pedagogical and didactical approaches

The following pedagogical and didactical approaches were used:

- comparative study of materials
- analysis of textbooks and other teaching materials
- direct observation of teacher training courses and classes of mathematics in lower secondary schools
- analysis of audio and video recording of individual lessons.

Taking into account the present social demands and the increasing presence of immigrant pupils, specific attention was paid to the best practices which support interdisciplinary and intercultural approaches to mathematics education.

Special interest was given to practices which emphasise a more situated approach to learning and strengthen links to real life and authentic activities in the educational process.

Didactic practices using IT as a pedagogical tool were taken into account as well.

Target groups

The project primary target group are mathematics teacher trainers in teacher training institutions of the partner countries.

The secondary target group are teacher trainees in mathematics at lower secondary level; these will be influenced by their trainers.

Some teacher trainers, provided with project materials and curricula units, tested them and proposed possible modifications to the project partners. This gave them the opportunity to reflect on their own teaching strategies and materials and thus increase their awareness and responsibility for their choices. Student teachers were stimulated to compare different didactical approaches and to have a critical view on the definition and selection of their own teaching methods.

All the above will have an implicit impact on their future pupils.

Main project outputs

- A set of didactic materials, based on selected best practices, to be used in lower secondary school mathematics teacher training, was produced, overcoming the difficulties related to the different training systems in the partner countries.
- The outlines of a possible common core of a curriculum for lower secondary school mathematics teacher training.

Both outputs are available on paper (in English) and in a DVD version (the languages used correspond to the partner countries – CZ-DK-FR-IT-SK – and include EN).

- A multilingual project web-site (in CZ-DA-EN-FR-IT-SK) [http://losstt-in-math.dm.unipi.it].

The outputs of the project were produced in co-operation with all partners, under the continuous co-ordination of the co-ordinator Institution. In particular, each partner selected best practices in its country and circulated them to other partners before their discussion, final selection and piloting. Each selected best practice was simultaneously piloted at its author/proponent's institution and at another partner institution at first, and at a third non-partner teacher training institution later. Each proposal was piloted in, at least, two different partner countries, thus allowing the project partners to have a more significant feedback.

The project web-site was designed and constructed under the direct responsibility of the co-ordinator after a common discussion on its structure.

Outputs addressee

- The didactic materials were made available to the Institutions with an accreditation in teacher training both in the partner and other European countries. Such dissemination made it possible to reach the primary and secondary target groups: mathematics teachers trainers and student-teachers in mathematics. Pupils in lower secondary schools are expected to benefit from the use of these didactical materials.
- The curricula overviews are primarily addressed to lower secondary school mathematics teacher trainers and their students in the partner and other European countries. These were delivered also to the educational policy makers.
- The web-site is available to the broad mathematics educational community.

Monitoring and evaluating of the project

The project was continuously monitored by the co-ordinator and each of the other project Working Team members.

The project Working Team meetings provided the opportunity for global monitoring of the activities so far carried out and their internal evaluation, step by step.

The external evaluation was required from Leo Rogers (United Kingdom), a prominent international expert in mathematics education; it was presented and discussed in specific evaluation meetings at the end of each of the first two years, before starting the new year activities, and during the third year, when all the materials were ready for the publishing (book and DVD).

Further opportunities for monitoring and externally evaluating the project activities were given by their presentations to the national and international community of researchers in Mathematics Education during Conferences and Workshops.

Dissemination

The data collection for the comparative study of lower secondary school mathematics teacher training systems and curricula required direct involvement of teacher training institutions in the partner countries.

The project partners undertook the specific task to keep continuously in contact with some mathematics teacher trainers in the partner countries and ask them to give evidence of good training practices. A few of these trainers were also asked to pilot the final collection of the best practices selected by the project Working Team.

After the final project outputs were widely distributed, the European teacher training institutions and their mathematical staff were finally asked to contribute to the piloting of the best of the above mentioned practices and to design their curricula according to the project final proposal.

The above activities therefore ensured continuous and direct information and dissemination of proposals, results and experiences of the project.

All outputs and products were made available to the public in the project web-site.

Long-term exploitation

The project web-site makes the project results, experiences and outputs easily available to the lower secondary school mathematics teacher trainers and their students involved in initial training (future mathematics teachers) across Europe, thanks to the fact that, besides the languages of the partner countries (CZ-DA-FR-IT-SK), English was used.

The co-ordinator and the project partners will keep disseminating all the project activities, the in-progress and final results by their presentations to the national and international community of teacher trainers and scholars in Mathematics Education during Conferences and Workshops, which they are used to attending for many years.

The above measures should ensure a long-term exploitation of the project results.

Project partnership composition

Centro di Ateneo di Formazione e Ricerca Educativa – CAFRE of the University of Pisa **(IT)** (*Project co-ordinator*) is the Centre in charge of research and training in the educational field. CAFRE members' expertise includes the organization and the management of the Specialization School for Secondary Teachers – SSIS of Tuscany. They are University of Pisa scholars active in different subjects and have experience in participation and co-ordination of national and European trans-national co-operation projects, including Socrates/Comenius 3.1 and 2.1 projects. [http://www.cafre.unipi.it]

***Skaarup Seminarium* (DK)** is the oldest public College of Education in Denmark. It educates teachers for primary and lower secondary schools. The College also provides short courses for qualified teachers and diploma studies in general didactics. The experience of the institution in international projects, both as co-ordinator and partner, is wide and includes Erasmus, Lingua, Comenius 2.1 and Tempus projects. [http://www.skaarupsem.dk]

***Institut Universitaire de Formation des Maîtres – IUFM of the Académie de Créteil* (FR)** is responsible for the training of primary and secondary schools teachers (of general, technological or professional subjects). The global amount of its students and trainees is approximately 5.000, including the primary, the general secondary and the technical and professional secondary degree schools. The IUFM partners are the nearby Universities and the schools of the Académie de Créteil. [http://www.creteil.iufm.fr]

***Department of Mathematics "Ulisse Dini" of the University of Florence* (IT)** is the main Department at the University working in mathematics and mathematics education. Many professors and researchers at the Department are responsible for the training of upper and lower secondary school mathematics teachers attending the courses of SSIS of Tuscany at the University of Florence. Members of the Department are also involved in both national research projects in mathematics and mathematics education and international projects action plans, such as Erasmus and Comenius. [http://www.math.unifi.it]

***Department of Mathematics and Computer Science of the University of Siena* (IT)** is in charge of the organization and management of mathematics courses offered by the University of Siena to the SSIS of Tuscany. The Department is a partner of several national and international projects. [http://www.dsmi.unisi.it]

***Faculty of Education of the Charles University in Prague* (CZ)** prepares teachers for all types and levels of schools, specialists and scientists in the area of pedagogy, educational psychology and didactics. Depending on the type of study, the Faculty of Education awards Bachelor, Master and Doctor Diplomas and Degrees. In the area of international co-operation, the Faculty of Education focuses on various types of projects in the Socrates programme (Comenius, Lingua, Grundtvig, Minerva, Arion, Erasmus). [http://www.pedf.cuni.cz]

***Matej Bel University* (SK)** is located in Banska Bystrica. The main focus of its Pedagogic Faculty is to prepare teachers for elementary schools and lower secondary schools. In recent years, however, it has also offered study programs in social and theological professions. The graduates are awarded Bachelor, Master and Doctoral degrees. Department of Mathematics has recently been involved in Socrates programme (Comenius 3.1). [http://www.pdf.umb.sk]

Acknowledgment

The project partners greatly benefited from the continuous monitoring by Leo Rogers (UK), mathematics education consultant and project evaluator. The suggestions, comments and remarks in his yearly reports have been very helpful for the project's achievement.

An acknowledgement goes to Giuseppe Fiorentino (IT), technology consultant for the project, who crafted the project's website and edited both this book and the DVD.

Thanks also to the translators Marta Hosková and Hana Moraová (CZ), Solveig Gaarsmand (DK), Christine Alves Smith (FR), Catia Mogetta (IT) and Iveta Dzúriková (SK).

Part I

Lower Secondary School Teacher Training Systems

1. COMPARISON OF SCHOOL SYSTEMS

Although the project's main concern is about teacher training, it could be useful to provide the reader with a general picture of the school systems in the project partner countries. In the table below the main focus is on the years of education at the different school levels.

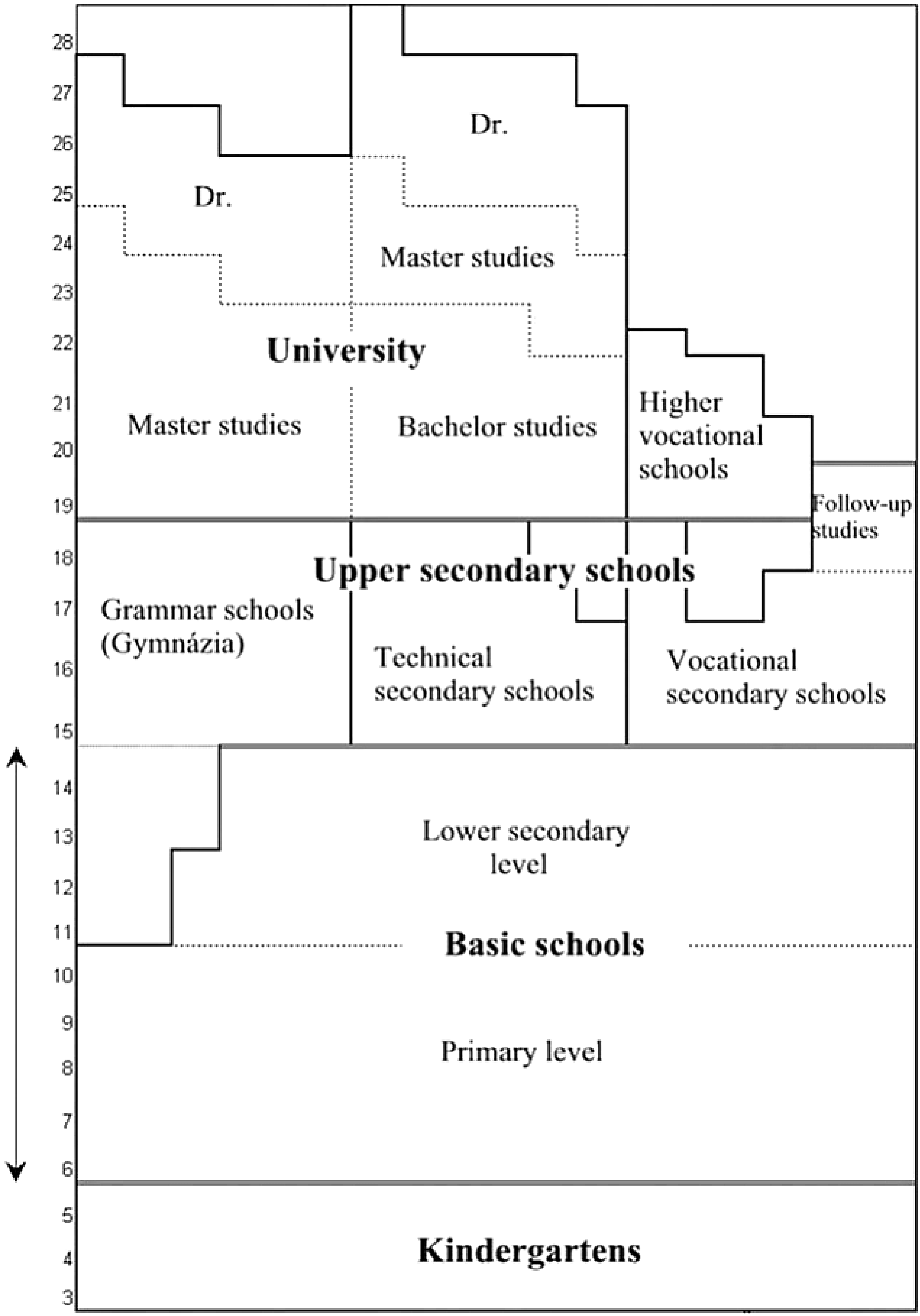
SCHOOL SYSTEMS (excluding infant school)								
Country	**Age at entrance**	**Compulsory education**		**Lower secondary school**		**Upper secondary school**		**Total years**
		Years	**Age range**	**Years**	**Age range**	**Years**	**Age range**	
CZ	6	9	6→15	4	11→15	4	15→19	13
DK	7	9[10]	7→16[17]	3[4]	13→16[17]	3	16[17]→19[20]	12[13]
FR	6[5]	9[10]	6[5]→15	4	11→15	3	15→18	12[13]
IT	6	9	6→15	3	11→14	5	14→19	13
SK	6	10	6→16	5	10→15	4	15→19	13

Table 1. School systems: age of entrance and duration.

Key: [ ] Age depending on the pupil's choice (see the pertinent national school system)

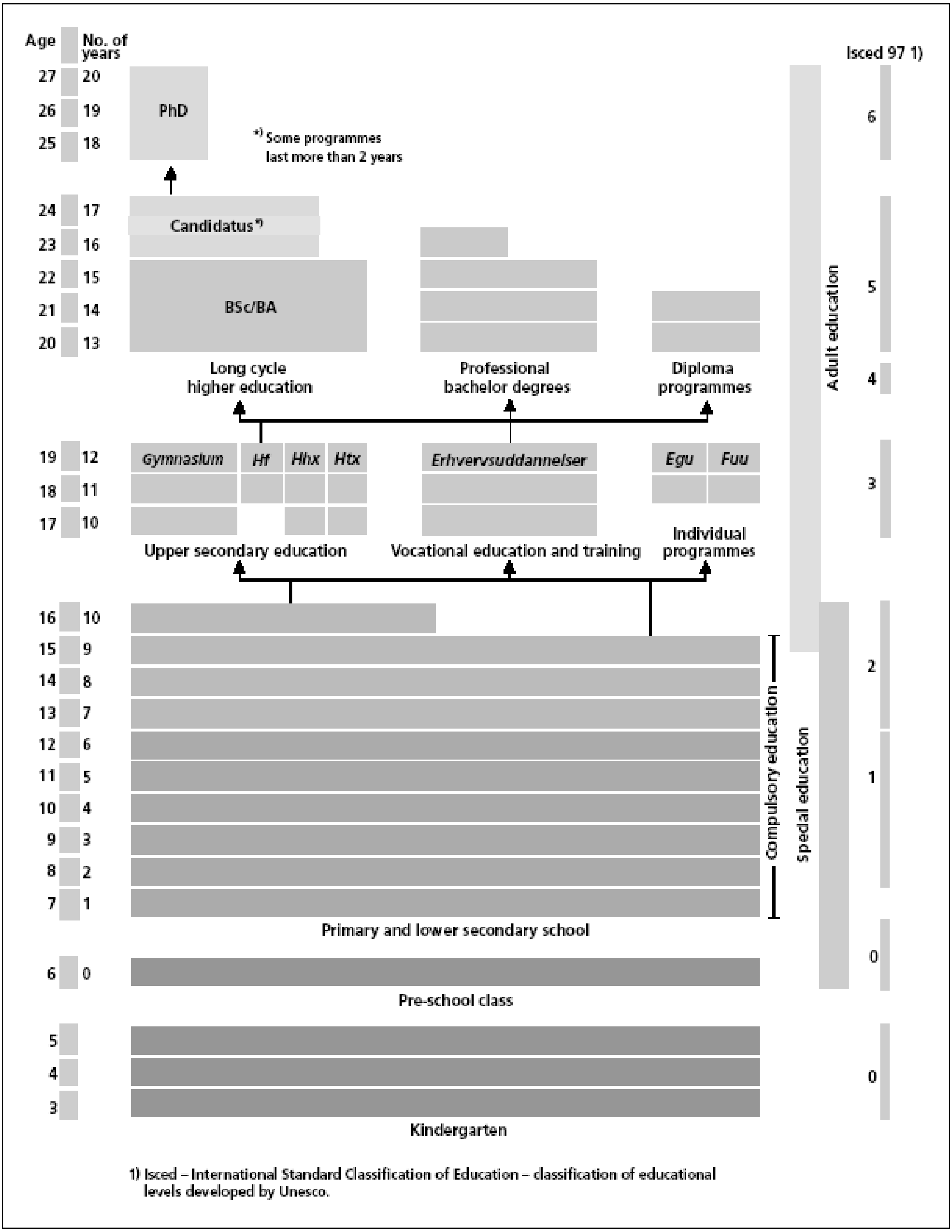
The schemes describing each school system, provided to the reader in the next pages, will make it possible to better understand both differences and commonalities.

CZECH EDUCATION SYSTEM



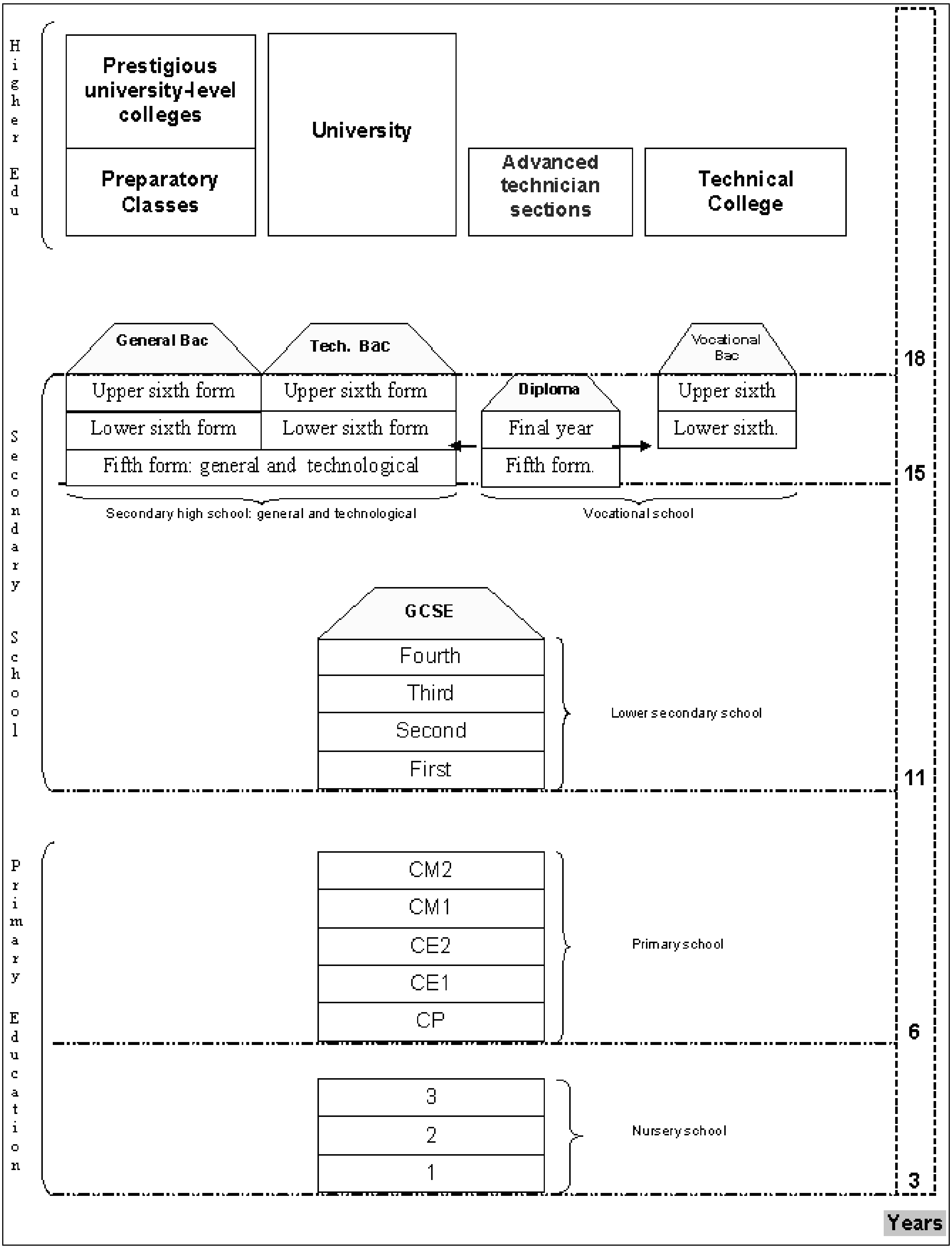
Picture 1. Czech Education System

DANISH EDUCATION SYSTEM



Picture 2. Danish Education System

FRENCH EDUCATION SYSTEM



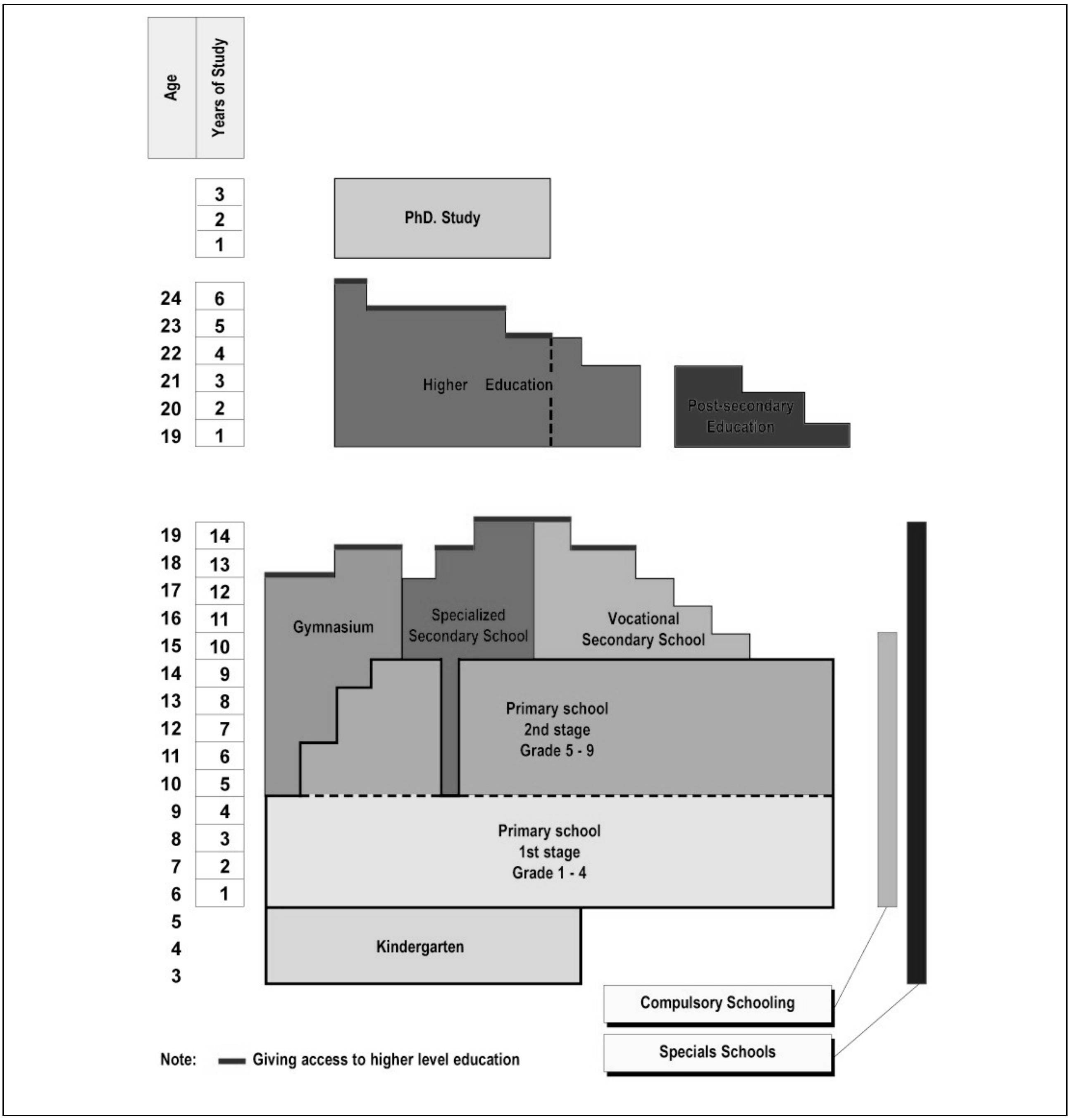
Picture 3. French Education System

ITALIAN EDUCATION SYSTEM

EDUCATIONAL INSTITUTION		Years	Age (option 1)	Age (option 2)
Higher education and training	*University* ***or*** *Higher technical training*	3 (Research doctorate) + 2 (Specialistic *laurea* degree) + 3 (*Laurea* degree)		
STATE EXAM				
Second cycle (upper secondary school 1st year is compulsory)	*Lyceum* ***or*** *Vocational school*	5 (2+2+1) 5 (3+1+1)	13,5 - 18,5	14 - 19
STATE EXAM				
First cycle	*First degree secondary school*	3 (2 + 1)	10,3 - 13,5	11 - 14
	Primary school	5 (1+2+2)	5,5 - 10,5	6 - 11
Infant school		3	2,5 - 5,5	3 - 6

Picture 4. Italian Education System

SLOVAK EDUCATION SYSTEM



Picture 5. Slovak Education System

2. COMPARISON OF LOWER SECONDARY SCHOOL MATHEMATICS STANDARDS

One of the project's aims is to show that, in Europe, it is possible to design a curriculum for lower secondary school mathematics teacher training that, despite the differences between teacher training systems throughout Europe, includes a relevant common set of topics to be introduced to trainees.

Before trying to show this fact, it is, however, necessary to investigate which are the standards for mathematics education at lower secondary school level in the different European countries. If they turn out to be quite different, any attempt to outline a European curriculum will be hard or even impossible. In this chapter standards accepted in the project's partner countries are compared.

The analysis of the tables in the sections below actually shows that, despite some differences between the input standards (that is the output primary schools standards), there is little difference, as expected, between the output lower secondary school mathematics standards accepted in the project's partner countries.

TOPICS	Czech Republic	Denmark	France	Italy	Slovak Republic
Number systems (operations incl.)					
Rational numbers	+	+	+	+	+
Fractions	+	+	+	+	+
Decimal numbers	+	+	+	+	+
Real numbers	+	+	+	+	+
Powers	+	+	+	+	+
Roots	+	+	+	+	+
Proportionality					
Percent	+	+	+	+	+
Ratio	+	+	+	+	+
Proportionality, rule of three	+	+	+	+	+
Divisibility					
Multiple and divisor	+	+	+	+	+
Prime numbers	+	+	-	+	+
GCD	+	-	+	+	+
LCM	+	-	-	+	+
Factorization	+	+	-	+	+
Expressions					
Numerical and algebraic expressions	+	+	+	+	+
Polynomials	+	+	-	+	+
Rational expressions	+	+	-	+	+

TOPICS	Czech Republic	Denmark	France	Italy	Slovak Republic
Equations, Inequalities					
Expressions	+	+	+	+	+
Linear equation	+	+	+	+	+
Quadratic equations	+	-	-	+	+
Linear inequalities		+	+	+	+
Systems of linear equations	+	+	+	-	+
Functions					
Coordinate system	+	(primary)	+	+	+
Properties of functions	+	+	-	-	+
Direct proportion	+	+	+	+	+
Indirect proportion	+	+	+	+	+
Linear function	+	+	+	+	+
Quadratic function	+	-	-	+	+
Trigonometric functions	+	-	-	-	+
Basic 2D notions					
Point, straight line, plane	+	+	+	+	+
Half-line, segment, half-plane, angle	+	+	+	+	+
Circle, circumference	+	+	+	+	+
Triangle, quadrilateral, polygon	+	+	+	+	+
Sets of points of given property	+	+	+	+	+
Trigonometry in rectangular triangle	+	-	+	+	+
Basic solids					
Polyhedron	+	-	+	+	+
Cube, cuboid, prism	+	+	+	+	+
Pyramid	+	+	+	+	+
Sphere, cylinder, cone	+	+	+	+	+
Geometric mappings					
Congruence of geometric figures	+	+	-	+	+
Similarity of geometric figures	+	+	-	+	+
Point symmetry, line reflection, translation	+	+	+	+	+
Constructions	+	+	+	+	+
Measurement	+	+	+	+	+

Table 2. Output lower secondary school mathematics standards

Further topics included in standards

The following topics are not explicitly mentioned in all standards, but they are implicitly present in all mathematical educational systems at the metacognitive[2] level. Except *Statistical probability* they belong to both content knowledge and metacognitive domain. The metacognitive level is not explicitly stated.

TOPICS	Czech Republic	Denmark	France	Italy	Slovak Republic
Data representation and organization	-	+	+	+	+
Estimates	-	+	+	+	-
Possibilities and limitations in using mathematics as a description and a basis for decisions	-	+	+	-	-
Statistical probability	-	+	-	-	+
Communication and problem solving	-	+	+	-	-

Table 3. Further topics included in standards

Metacognition combines various attended thinking and reflective processes. It can be divided into five primary components: (1) preparing and planning for learning, (2) selecting and using learning strategies, (3) monitoring strategy use, (4) orchestrating various strategies, and (5) evaluating strategy use and learning. Teachers should model strategies for learners to follow in all five areas. To be effective, metacognitive instruction should explicitly teach students a variety of learning strategies and also when to use them[3].

Some theories underpinning learning strategy research:

- O'Malley and Chamot (1990)[4] classify strategies in the following way:
    - cognitive strategies;
    - metacognitive strategies;
    - social strategies;
    - affective strategies.

- Rebecca Oxford (1990)[5] distinguishes:
    - Direct Strategies (memorizing, cognitive processing, compensation);
    - Indirect Strategies (metacognitive, social and affective).

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[2] In education, the term *metacognition* can be defined as "awareness of one's own knowledge or problem-solving abilities".

[3] Anderson, J. (2002). The Role of Metacognition in Second Language Teaching and Learning. Available at http://www.cal.org/resources/digest/0110anderson.html

[4] O'Malley, J.M. & Chamot, A.U. (1990). *Learning Strategies in Second Language Acquisition*. Cambridge University Press.

[5] Oxford, R.L. (1990). *Language Learning Strategies: What Every Teacher Should Know*. Newbury House.

Examples of metacognitive strategies

- *Problem identification*: Explicitly identifying the central points needing resolution in a task or identifying an aspect of the task that hinders the successful completion.
- *Self-management*: Understanding and arranging for the conditions that help successfully accomplish the task.
- *Self-monitoring*: Checking, verifying or correcting one's comprehension or performance in the course of problem solving.

OUTPUT PRIMARY SCHOOL MATHEMATICS STANDARDS
INPUT FOR LOWER SECONDARY LEVEL

The following topics are present in all five partner countries:

- Natural numbers.
- Number line.
- Position systems.
- Rounding natural numbers.
- The four basic arithmetical operations.
- Basic geometric conceptions.
- Construction of simple regular geometric figures.
- Drawing and the use of pictures.
- Working with experiment and investigation.
- Solving word problems.

Further topics

Czech Republic

- Decimal numbers with tenths and hundreds, their addition and subtraction, their use in concrete models (Money, weight, length).
- Fractions as a part of a whole, representation of fractions, simple cases of addition of fractions with the same denominator.
- Units of length, units of area mm^2, cm^2, dm^2, m^2, ha.
- Recognition of 3D-figures (cube, cuboid, prism, cylinder, pyramid, cone, sphere); nets of cuboids and cubes.
- Calculation of the surface area of a cuboid and cube by adding the area of their bases and faces.

Denmark

- Whole numbers.
- Examples how to use variables, including formulas, simple equations and functions.
- Percent and the use of percent in everyday life.
- Coordinate system including the connection between number and drawing.

France

- Decimal numbers with tenths, hundreds and thousands, their addition, subtraction and multiplication.
- Division of a natural number by two-digit numbers.
- Units of weight, length, area and capacity.
- Fraction as a part of a whole, representation of fractions.
- Recognition of 3D-figures (cube, cuboid); nets of cuboids and cubes.
- Geometric mappings: rotation.

Italy

- Decimal numbers, their addition and subtraction, their use in concrete models (money, weight, length).
- Fractions as a part of a whole, representation of fractions.
- Units of length, units of area mm^2, cm^2, dm^2, m^2, ha.
- Determination of the area of simple polygons (from the triangle to the regular exagon) using formulas.
- Solving word problems resulting in simple calculations of perimeters and areas of simple polygons.
- Measures by means of the measure international system, converting them into each other in real simple cases.
- Determination in simple cases of lengths, areas and volumes.
- Data collection of quantities.
- Histograms.

Slovak Republic

- Decimal numbers with tenths, hundreds and thousands, their addition, subtraction and multiplication and their use in concrete models (money, weight, length).
- Division of natural numbers by two-digit numbers.
- Fractions as a part of a whole, representation of fractions, addition of fractions.
- Units of weight, length, area and capacity.
- Determination of the area of simple polygons using formulas.
- Solving word problems resulting in simple calculations of perimeters and areas of simple polygons.
- Recognition of 3D-figures; nets of cuboids and cubes; calculation of the surface area and volumes of cuboids and cubes.
- Elements of algebra (variable, equation, inequality).

CZECH LOWER SECONDARY SCHOOL MATHEMATICS STANDARDS

Knowledge	Abilities
Divisibility • Multiples and divisors of a number; • Prime numbers; • Least common multiple, greatest common divisor; • Powers of natural numbers; • Integer numbers.	• To decide whether a given number is/is not a multiple/divisor of a certain number. • To find all divisors of a given number. • To decide without dividing if the number is divisible by 2, 3, 4, 5, 6, 8, 9, 10. • To decompose a natural numbers into its prime divisors. • To find the LCM and GCD of a group of numbers.
Number systems • Real numbers; • Absolute value; • Rational numbers; • Fractions; • Operations with rational numbers; • Powers; • Roots.	• To recognize a natural, integer, rational, irrational number. • To write numbers in the decimal system • To represent all rational and some irrational numbers on the number line. • To understand the notions positive, negative, non-positive, non-negative numbers. • Find the absolute value of a real number. • To define a rational number. • To compare rational numbers. • To write a fraction as a decimal or periodic number. • To understand that one rational number might be expressed as infinitely many fractions. • To reduce and raise a fraction, simplify a fraction; to find irreducible fractions. • To make calculations with mixed numbers. • To make calculations with rational numbers using different methods. • To calculate second and third powers of a number. • To make calculations with powers. • To calculate square roots of some numbers, to find second and cube roots using tables and calculators.
Percents, ratio, proportion • Percent; • Ratio; • Proportion, rule of three.	• To calculate the percent as 1/100 of a whole • To make calculations about percents. • To apply percents - simple interest. • To express the ratio in its simplest form by cancellation. • To understand the progressive ratio as a short form of simple ratios. • To divide a number in a given progressive ratio. • To understand proportion as the equality of two ratios. • To calculate the unknown term of a proportion. • To apply the rule of three when solving problems.

Knowledge	Abilities
Expressions • Expressions; • Polynomials; • Rational expressions.	• To recognize expressions. • To construct expression. • To find the value of an expression for a given value of variable. • To apply the notions member, coefficient, degree, value of a polynomial. • To add, subtract, multiply polynomials, to divide a polynomial by a linear binomial. • To factorize polynomials. • To use formulas for $(A + B)^2$, $A^2 - B^2$, $A^3 - B^3$, $A^3 + B^3$. • To distinguish between polynomials and rational expressions. • To reduce and raise rational expressions by a number, monomial and linear binomial. • To make calculations with rational expressions.
Equations • Equations and their rearranging; • Linear equations; • Quadratic equations; • Systems of linear equations.	• To recognize equations. • To verify if a given object is a root of the given equation. • To define a root of an equation. • To add the same expression to both sides of an equation, to multiply both sides of an equation by the same non-zero expression. • To determine whether a given equation is linear. • To solve linear equations. • To classify the number of roots of a linear equation. • To define a root of a linear equation as the intersection on the graph of the corresponding linear function and x-axis. • To transform an equation with the unknown in the denominator into a linear equation. • To apply linear equations in problem solving. • To solve quadratic equations $ax^2 + b = 0$, $x^2 - m = 0, m > 0$. • To recognize systems of linear equations. • To understand that the solution of a system of two linear equations with two unknowns is an ordered number pair. • To have the idea about the number of solutions of a system of two linear equations with two unknowns. • To solve a system of two linear equations with two unknowns. • To find the solution of two linear equations with two unknowns as coordinates of the intersection of graphs of two linear functions.

Knowledge	Abilities
Functions • Coordinate system; • Functions; • Direct proportion; • Indirect proportion; • Linear function; • Quadratic function; • Trigonometric functions.	• To choose an appropriate coordinate system in a plane. • To represent a point in a given coordinate system. • To determine coordinates of a point represented in a given coordinate system. • To recognize a function given by a table, graph and formula. • To decide if a number belongs to the range of a function. • To find the range of a function from its graph, formula, table. • To find the function value for a given element from its range. • To decide if a given set of points is a graph of a function. • To decide from the graph if the function is increasing/decreasing. • To select functions that are direct/indirect proportions. • To find the coefficient of a given direct/indirect proportion. • To construct a graph of a direct/indirect proportion. • To select functions which are linear. • To construct a graph of a linear function. • To select functions which are quadratic of the form $y = ax^2$, $a > 0$. • To construct a graph of a quadratic function. • To define tangent, sine, cosine of an acute angle. • To grasp tangent, sine, cosine as dependencies, to sketch their graphs. • To find values of trigonometric functions using table and calculators. • To find α when knowing $\sin \alpha$, $\cos \alpha$, $\operatorname{tg} \alpha$. • To use trigonometric functions in problem solving.
Basic 2D-notions • Point, straight line, plane; • Half-line, line segment, half-plane, angle; • Sets of a given property; • Circle, circumference of a circle; • Polygon; • Triangle; • Quadrilateral.	• To know what defines a straight line in a plane. • To characterize the mutual position of 3 points/2 lines in a plane. • To find out if two lines are parallel, perpendicular. • To understand properties of half-lines, line segments, angles. • To measure an angle, classify angles by their size. • To know properties of an angle bisector. • To know and determine properties of adjacent angles, vertically opposite angles, alternate angles, corresponding angles on parallel lines. • To find simple sets of points of a given property. • To know and apply Thales' Theorem. • To define a circle, circumference of a circle. • To know the notion of an arc, to determine its length. • To characterize the mutual position of a circle and line. • To know properties of a tangent and of a secant to a circle.

Knowledge	Abilities
Basic 2D-notions (cont.)	• To know formulas for the perimeter and area of a circle. • To recognize a polygon. • To decide if a given polygon is convex or concave. • To characterize a regular polygon. • To know and be able to use properties of triangles. • To classify triangles. • To know properties of angles, lines joining the mid-points of 2 sides, heights, medians of a triangle. • To characterize the circumscribed circle, inscribed circle. • To calculate the height and area of a triangle. • To describe an isosceles and equilateral triangle. • To know the Theorem of Pythagoras and to apply it when solving problems. • To recognize a quadrilateral. • To calculate the area of a quadrilateral by dividing it into two triangles. • To know the sum of inner angles of a quadrilateral. • To classify quadrilaterals. • To describe trapezium and its special types, parallelogram, rhombus, rectangle, square. • To calculate and to apply formulas for the area of a trapezium, parallelogram, rhombus, rectangle, square.
Geometric mappings • Congruence of geometric figures; • Similarity of geometric figures; • Isometries; • Point symmetry; • Line reflection; • Translation.	• To understand intuitively the concept of the congruence/similarity of figures. • To have an active command of theorems conditions for the congruence/similarity of triangles. • To understand the notions object, image, to know that the image of congruent figures in an isometry are congruent figures, to understand the notion of a fixed point. • To find the image of a point, line, triangle and circle in an isometry. • To define a point symmetry/line reflection/translation, to know their properties. • To recognize figures which are isometric, • To apply symmetries when solving problems.
Constructions • Basic geometric constructions; • Constructions of an angle; • Construction of a triangle; • Construction of a quadrilateral; • Construction of a regular polygon; • Construction of a circle.	

Knowledge	Abilities
Polyhedra • Polyhedron; • Cube; • Cuboid; • Prism; • Pyramid.	• To recognize polyhedra. • To describe their vertices, edges, faces on a model. • To define and recognize a cube/cuboid, to sketch it in an oblique projection. • To know its properties. • To mark its face and body diagonals. • To know and apply formulas for a cube/cuboid volume and surface area. • To know the properties and be able to sketch a prism/a pyramid. • To describe its basis and lateral surface. • To use Theorem of Pythagoras and trigonometric functions for determining the height and lengths of edges. • To sketch nets.

Table 4. Czech lower secondary school mathematics standards

CZECH INPUT SITUATION
SUMMARY OF MATHEMATICS TAUGHT AT THE PRIMARY LEVEL

Arithmetic

- Natural numbers and their notation in the decimal system.
- Representation on a number line.
- The number zero.
- Comparison of natural numbers.
- Rounding natural numbers.
- Operations with natural numbers including division by one- and two-digit divisors.
- Solving word problems resulting in one or two operations with natural numbers.
- Decimal numbers with tenths and hundreds, their addition and subtraction.
- Using decimal numbers in concrete models (money, weight, length).
- Fraction as a part of a whole, representation of fractions.
- Simple cases of addition of fractions with the same denominator.

Geometry

- Recognition of 2D-figures (triangle, rectangle, square, quadrilateral, circle).
- Measuring lengths of line segments - units of length mm, cm, dm, m, km and their conversions.
- Construction of line segments of given lengths.
- Perimeter of a figure.
- Construction of lines parallel to a given line, perpendicular to a given line[6].

[6] In Czech schools, special triangles are used for these constructions - isosceles right-angled triangles with the height to the base.

- Construction of a circle with a given center and radius.
- Construction of a rectangle, square and right-angles triangle.
- Determination of the area of a rectangle using the square grid.
- Calculation of the area of a circle, rectangle.
- Units of area mm^2, cm^2, dm^2, m^2, ha.
- Recognition of 3D-figures (cube, cuboid, prism, cylinder, pyramid, cone, sphere).
- Nets of cuboids and cubes.
- Calculation of the surface area of a cuboid and cube by adding the area of their bases and faces.
- Solving word problems resulting in simple calculations of perimeters and areas of rectangles and squares.

DANISH LOWER SECONDARY SCHOOL MATHEMATICS STANDARDS

Numbers and algebra

Knowledge	Abilities
Rational numbers and the expansion to real numbers	Working investigating, especially with systematic counting and with the numbers mutual size as a part of forming a general understanding of numbers
Cultural-historical meaning of numbers as a means of description.	Using mental arithmetic and estimate
	Using pocket calculator and computer in problem solving
	Understand and using formulas for example in calculation of interest and volume
	Understand and using expressions containing variables
Percent	Using percent
	Calculating with fractions including solving of equalities and algebraic problems.
	Investigate and describe "changes" and structures for example in series of numbers, series of figures, and patterns
Conception of functions	Using graphical methods in solving equalities and systems of equalities
	Solving simple equalities and solving inequalities by inspection.

Geometry

Knowledge	Abilities
Properties of different geometric figures	Using the properties of different geometric figures
	Drawing pictures after given preconditions
	Using basic geometric conceptions including ratio of size and the mutual position of lines
	Understand and produce: • workshop drawings • isometric drawings • perspective drawings in description of the surroundings Investigate, describe and estimate connections between a drawing and the object drawn
The conception of measurement	Using the conception of measurement, including measuring and calculation of circumference, surface and volume
Scale, similarity and congruence	Using scale, similarity and congruence
	Execute simple geometric calculations for example using Pythagoras` theorem
	Working with simple geometric proofs
	Using computers in drawing, investigations and calculations concerning geometric figures

Applied mathematics

Knowledge	Abilities
	Choose arithmetical operations Use percent Calculate with proportion in various connections
	Handle examples of problems in societal development with respect to: Economy, technology and environment
	Make economic considerations regarding daily purchase, transport, housing conditions, salary and calculation of tax
	Working with interest and making calculation of interest especially regarding saving up, borrowing, buying on credit
	Work with and investigate formulas and functions entering mathematic models
Possibilities and limitations in using mathematics as a description and a basis for decisions	Working with statistical description of a collection of data, with the emphasis on method and interpretation
	Execute simulations for example by the computer
Statistical probability	Using computers for calculations, simulations, investigations and descriptions included societal conditions
	Using mathematics as a tool in solving practical and theoretical problems in a comprehensive way

Communication and problem solving

Knowledge	Abilities
	Understand and take a position on information's, with mathematic expressions
	Formulate problems, describe procedure and give solutions in a understandable manner, both in writing and oral
	Using experimenting and investigating in working and formulating results of the professional knowledge, which is obtained
	Choose suitable professional methods, working and tool by solution of transverse problems
	Cooperation with others in mathematic problem solving
	Using systematizing and mathematic argumentation
	Using variables and symbols, when rules and connections are going to be proved
	Using geometric drawings in formulating hypotheses and argumentation
	Understand that choosing a specific mathematic model can mirror a specific norm of value
	Alternate between practical and theoretical considerations in solutions of mathematic problems

Table 5. Danish lower secondary school mathematics standards

DANISH INPUT SITUATION
SUMMARY OF MATHEMATICS TAUGHT AT THE PRIMARY LEVEL

Numbers and algebra

Knowledge	Abilities
Whole numbers Decimal numbers Fractions	
	Using experiences from everyday life in connection with working with numbers in school
System of numbers Number line Position systems The four basic arithmetical operations	Use mental arithmetic and estimate
	Using pocket calculator and computer in problem solving
	Working with counting's and examples on connections and rules for the four basic arithmetical operations
Examples of how to use variables, including formulas, simple equalities and functions	
Percent and the use of percent in everyday life	Calculation with decimal numbers and using fractions linked to percent and concrete connections
Coordinate system including the connection between number and drawing	Working with "changes" and structures for example in series of numbers, series of figures, and patterns

Geometry

Knowledge	Abilities
	Using geometry in the description of object from everyday life including figures and patterns
	Investigate and describe simple figures in plane
Basic geometric conceptions like angles and parallelism	
	Working with real models and simple drawing of these
The method of different cultures to give depth in a picture	Investigate the application of drawing methods
	Measure and calculate perimeter, area and volume in concrete situations
	Draw, investigate and do experimentation in relation to geometric figures, including using computers

Applied mathematics

Knowledge	Abilities
	Choose and use arithmetical operations in various connections
	Using professional tools included numbers, graphical representation and statistics in the solution of mathematical problems concerning everyday life, family life and the nearest societal
	Working with simple percent calculations including discount price
	Describe and interpret data in tables and diagrams
	Collect and treat data and make simulations for example with a computer
	Experimentation with chance

Communication and problem solving

Knowledge	Abilities
Working with experiment and investigation	Make hypotheses and using "trial and error" in constructing professional concepts
	Formulate, solve and describe problems and in this connection choose different methods
	Cooperate with others in applying mathematics in problem solving
	In relation to mathematics investigate, systematize, and argue using concrete materials

Table 6. Summary of mathematics taught at the primary level

FRENCH LOWER SECONDARY SCHOOL MATHEMATICS STANDARDS

Mathematics - Year 1 (sixième)

Knowledge	Abilities
Magnitudes and measurements • Length of a circle. • Perimeter and surface of a rectangle. • Surface of a right-angled triangle. • Volume of a right-angled parallelepiped starting from a paving.	Geometrical and numerical work can constitute a privileged ground to approach reasoning on circumscribed deductive small islands, especially in connection with lengths and surfaces.

Numbers and numerical calculation
• Decimal writing and operations +, -, ×.
• Division by a whole number:
  o quotient and remainder in Euclidean division,
  o approximate division.
• Truncation and round fractional
• Writing of the quotient of two whole numbers, simplifications.
	Pupils are familiar with integer numbers but not with decimal numbers. It is necessary to work on the significance of decimal writing and to link this with work on operations and the multiplication and division by 0.1, 0.01 or 0.001. Writing fractions appeared only in very simple examples in primary school. It is now necessary to link this with decimal writing.
	Multiplication of two decimal numbers.

Literal algebraic calculation • Substitution of numerical values for letters in a formula.	
Numerical functions • Changes of surface units and of length units. • Application of a rate of percentage. • Study of examples involving or not proportionality.	To know how to approach multiplicative situations whose treatment makes it possible to use or to highlight the properties of linearity or the presence of a proportionality factor.
Representation and organization of data • Examples to read and draw up tables and graphs.	
Configurations, constructions and transformations. • Ring. • Particular triangles, triangles. • Rectangle, rhombus. • Transformation of figures by axial symmetry. • Right-angled parallelepiped.	It is necessary to develop training which binds pupils to the demonstration. Preparation of deductive reasoning in particular by taking into account information in diverse forms.
Locations, distances and angles. • Positive X-coordinates on a graduated line. • Location by relative integers, on a graduated line (abscissa) and in the plane (with coordinates).	

Table 7. Mathematics at lower secondary school in France - Year 1

Mathematics - Years 2 -3 (5ème-4ème)

Knowledge	Abilities
Magnitudes and measurements • Measurements of time. • Sum of the angles of a triangle. • Area of a parallelogram, a triangle, a disc. • Side surface and volume of a right prism, a cross-section cylinder. • Quotients of usual magnitudes. • Volume of a pyramid. • Volume and side surface of a circular cone.	
Numbers and numerical calculation • Sequence of calculations, operational priorities. • Product of two fractions. • Comparison, sum and difference of two fractions with equal or multiple denominators. • Comparison, sum and difference of relative numbers in decimal writing. • Operations (+, -, ×, :) on relative numbers in decimal or fractional form (not necessarily simplified). • Powers of relative numbers and exponent. • Scientific notation of the numbers. • Keys √ and cos of a calculator, *opposite*.	In the third year, control of the four operations with decimal numbers, and with fractional numbers. Multiples and dividers, criteria of divisibility. The control of the procedures is acquired through activity, especially the resolution of problems. To distinguish the nature of numbers: exact numbers, values displayed on a computer screen, values approached with a given precision. To practise the various operations: without calculator, mentally and with a machine. Mental calculation allows consolidation of knowledge, for instance multiplication tables, as well as control of use of a computer by determining the order of magnitude of a result.

Literal calculation
• Equalities:
    o k (a + b) = ka + kb
    o k (a - b) = ka - kb
• Test of an equality or an inequality by substitution of numerical values for one or more variables.
• Expansion of expressions.
• Effect of the addition and the multiplication on the order.
• Simple linear equations with one unknown.
	In the second year, substitution of numbers for letters makes it possible to carry out numerical calculations, to include/understand and control the writing rules of literal algebraic expressions.
	Resolution of equations and inequalities, conservation of equalities, inequalities. The distributivity of the multiplication compared to the addition is not limited to numerical examples; it leads to a first contact with the concept of identity. Development of expressions, concepts of numerical functions

Mean velocity • Calculations with percentages. • Changes of units for quotients of usual magnitudes. • Applications of proportionality.	Examples, counterexamples, particular cases in opposition to the general case. Initiation in reasoning using counter-examples.
Representation and organization of data • Classes, organisation of a statistical distribution.	Step consisting in synthesizing in numerical or graphic form the information collected on all the elements of a population.

<table>
<tr><th>Knowledge</th><th>Abilities</th></tr>
<tr><td>Representation and organization of data (cont.)
• Frequencies.
• Bar charts, pie-charts.
• Cumulative distributions. Frequencies.
• Averages.
• Starting the use of spreadsheet-graphics packages.</td><td>Study of statistical statements in the form of tables or graphs while being interested in the relevance of the choice of the classes and the mode of chart selected. Distinction between the case where one has data on the whole population and the case where the data relate to a grouping of the population in class intervals. Utilisation of spreadsheets graphics packages making it possible to undertake in experiments the search for a distribution in classes adapted to a problem arising by visualizing the various paces of the associated diagrams.</td></tr>
<tr><td>Configurations, constructions and transformations.
• Construction of triangles (with instruments and/or geometrical software).
• Centroid of a triangle. Parallelogram.
• Transformation of figures by central symmetry.
• Right prisms, cylinders of cross-section.
• Triangle: Theorems relating to the midpoints of two sides.
• Triangles determined by two parallel straight lines cutting two secants: proportionality of lengths.
• Orthocenter of a triangle.
• Right-angled triangle and its circumscribed circle.
• Transformation of figures by translation.
• Pyramid, circular cone.</td><td>In solid geometry, construction of solids starting from development of these solids. In plane configurations and geometrical transformations: it aims to obtain on geometrical objects a look which comes from evolution of points of view, confrontation of configurations represented by geometrical figures. Using geometrical transformations is a step to be got for all technical or scientific uses. Gradual passage from a vision of geometrical figures to one of the whole plane. Translation is appropriate to mark such an evolution. Conservation of alignments, distances and angles by a translation.
It is the composition of different translations which will make useful the introduction of the vectors.
In Thales triangle situation, all the useful results of proportionalities are presented starting from the situation obtained by having two secants “cut” by two parallels.</td></tr>
<tr><td>Locations, distances and angles.
• Location on a graduated line.
• Distance between two points.
• Location in the plane (coordinates).
• Triangular Inequality.
• Graphic relation of proportionality: representation.
• Pythagoras' theorem and its reciprocal.
• Distance from a point on a line;
• Tangency to a circle.
• Cosine of an acute angle.</td><td>One can speak of a point of abscissa, say, -4/3 on a graduated line, as in the second year, it will then be located with an approximation of the quotient, placement of a point using a construction resulting from Thales' theorem is considered in the final year.
Triangular inequality appears from the second year. The construction of triangles leads naturally to it. It means simply to know, for example, that it is useless to try to build a triangle whose side lengths are respectively 10 cm, 15 cm, 26 cm , that a triangle whose sides lengths are 11 cm, 15 cm and 26 cm will be flattened.</td></tr>
</table>

Table 8. Mathematics at lower secondary school in France - Years 2 -3

Mathematics - Year 4 (3ème)

Knowledge	Abilities
Magnitudes and measurements • Usual magnitudes and compound magnitudes. • Surface area of the sphere, volume of the sphere	In a situation using magnitudes, one of the magnitude being a function of the other one. To represent graphically the situation in an exact way if that is possible, in an approximate way if not. Read and interpret such a representation. To know and use the fact that in an enlargement or a reduction of ratio k: • the surface area of a surface is multiplied by k^2 • the volume of a solid is multiplied by k^3
Numbers and numerical calculation • Calculations involving radicals. • Simple examples of algorithms (successive differences, Euclide...). • Numerical applications with a calculator. • Irreducible fractions.	To determine if two natural integers have no common divisor, but 1. To simplify a fraction to make it irreducible. To know that if a indicates a positive number $\sqrt{a}$ is the positive number whose square is a. To use equalities with numerical examples where a is a positive number: $(\sqrt{a})^2 = a = \sqrt{a^2}$. With numerical examples to determine the number x such as $x^2 = a$ where a indicates a positive number. To use the equalities: $\sqrt{(ab)} = \sqrt{a}\,\sqrt{b}$ and $\sqrt{(a/b)} = \sqrt{a}/\sqrt{b}$ with numerical examples, where a and b are two positive numbers
Literal algebraic calculation • Factorization (identities). • Problems that can be reduced to first degree equations. Inequalities. • Linear systems with 2 unknowns.	To factorize expressions such as $(x+1)(x+2)-5(x+2)$; $(2x+1)+(2x+1)(x+3)$ Knowing the equalities: • $(a + b)(a - b) = a^2 - b^2$ • $(a + b)^2 = a^2 + 2ab + b^2$ • $(a - b)^2 = a^2 - 2ab + b^2$ and using them on numerical and simple literal expressions such as: • $101^2 = (100+1)^2 = 100^2 + 200 + 1$ • $(x+5)^2 - 4 = (x+5+2)(x+5-2)$ To use the fact that the relative numbers ac and bc are in the same order as a and b if c is strictly positive, in the opposite order if c is strictly negative. To solve an equation such as: $ax + b = cx + d$. To determine an equation and to solve a problem leading to an equation, an inequality or a system of two simple equations.
Numerical functions • Linear functions and affine functions.	To know the notation $x \rightarrow ax$ for a fixed numerical value of a. To determine the algebraic expression of a linear function starting from the data of a number and its image. To represent a linear function graphically. On the graph of a linear function, to find the image of a given number, and to read the number from the image of this number.

<table>
<tr><th>Knowledge</th><th>Abilities</th></tr>
<tr><td>Numerical functions (cont.)</td><td>To know the notation x→ax+b for fixed numerical values of a and b.
To determine an affine function defined by data of two numbers and their images. To represent an affine function graphically.
On the graph of an affine function, to find the image of a given number and to read the number from the image of this number.</td></tr>
<tr><td>Representation and organisation of data
Representation and organisation of data
Characteristics of position of a statistical series. To approach characteristics of dispersion of a statistical series.
Initiation in the use of spreadsheets graphics packages in statistics.</td><td>A statistical series being given (in the form of list or table, or by a chart), to propose a median value of this series and to give its significance.
A statistical series being given, to determine its extent or that one of a given part of this series.</td></tr>
<tr><td>Configurations, constructions and transformations.
Solid geometry.
Sphere.
Problems of plane sections of solids. Properties of Thalès.
Vectors and translation.
Vector equalities.
Composition of 2 translations.
Composition of two central symmetries rotation, angle, regular polygon, images of figure by a rotation, regular polygon and inscribed angle.</td><td>To know that the section of a sphere by a plane is a circle. To know how to place the centre of this circle and how to calculate its radius knowing the radius of the sphere and the distance of the plane from the centre of the sphere. To know the nature of the sections of the cube, the right-angled parallelepiped by a plane parallel to a face, or an edge. To know the nature of the sections of the cylinder of cross-section by a plane parallel or perpendicular to its axis. To represent and determine the sections of a circular cone and a pyramid by a plane parallel to the base. To know and use in the right-angled triangle the relations between the cosine, the sine or the tangent of an acute angle and the lengths of two sides of the triangle.
To use the vectorial equality:
vect (AB) + vect(BC) = vect(AC)
and to connect it to two successive translations.
To build a representation of the vector sum using a parallelogram. To know and to use the vector writing. To express that the translation which transforms A into B transforms also C into D. To link this vector writing to a flattened parallelogram ABCD.
To know the image by a given rotation of a point, a circle, a line, a segment, a ray. To build an equilateral triangle, a square, a regular hexagon knowing its centre and a top.
To know and use in a given situation the two following theorems “ d and d’ are two secant lines in A. B and M are two points of d , distinct from A, C and N are two points of d’, distinct from A. If lines (BC) and (MN) are parallel then

AM/AB = AN/AC = MN/BC</td></tr>
</table>

Knowledge	Abilities
Configurations, constructions and transformations (cont.).	Are *d* and *d*' and two secant lines in A. Are B and M two points of *d*, distinct from A. Are C and N two points of *d* ', distinct from A. If AM/AB = AN/AC and the points A, B, M and points A, C, N are in the same order, then lines (BC) and (MN) are parallel. To compare an inscribed angle and the angle with the center intercepting the same arc.
Locations, distances and angles. Right angled triangle, trigonometric relations, and distance between two points in an orthonormal coordinate system of the plane. Coordinates of a vector in the plane in an orthonormal coordinate system. Sum of two coordinates of two vectors in the plane in an orthonormal coordinate system.	

Table 9. Mathematics at lower secondary school in France - Year 4

FRENCH INPUT SITUATION
SUMMARY OF MATHEMATICS TAUGHT AT THE PRIMARY LEVEL

General abilities - Cycle 3

In primary school as in lower secondary school, solving problems are at the centre of mathematical activities of pupils.

The two curricula emphasize the presentation on the same aims and suggest similar abilities, for example:

- abilities to search, to abstract, to reason, to prove at cycle 3 of primary school and abilities of reasoning: observation, analysis, to reason deductively at first year of lower secondary school («sixième»);
- to make assumptions and to test them at cycle 3 and to conjecture about a result at «sixième»;
- to argue about validity of a solution at cycle 3 and to develop an argument at «sixième»;
- to verify results and to put into words of the problem an answer at cycle 3 and to verify the results and to estimate their relevance according to the problem at «sixième».

Curricula of Cycle 3

Exploitation of numerical data

- problems solved with knowledge about positive natural and decimal numbers, and with studied operations;
- problems relating to proportionality, solved with a personal and appropriate reasoning;
- use of data organized by lists, by tables or represented by pie or bar charts or by graphics.

Knowledge of natural numbers

- decimal numeration: value of figures according to their position, series of numbers;
- written (by figures and by letters) and oral designations of numbers;
- to compare and to put in order numbers, to set numbers on a graduate line;
- arithmetical connections between numbers: double, half, quadruple, quarter, triple, third…., notably between numbers of common use, notion of multiple (multiples of 2, 5 and 10).

Knowledge of simple fractions and decimal numbers

- simple fractions: use, writing, to surround by two successive natural numbers, writing with a natural number and a fraction less than 1;
- decimal numbers: use, value of figures according to their positions in a writing with a decimal point, transition from writing with a decimal point to a fractional writing (decimal fractions) and vice versa, series of decimal numbers, connection between oral and writings with figures designations;
- comparison, to put in order, insertion, to surround decimal numbers, to set on a graduate line;
- approximate value of a decimal number, to within one unit, one tenth, one hundredth.

Calculus

- memorizing results about natural and decimal numbers;
- techniques of operations: addition, subtraction with whole or decimal numbers, multiplication with two natural numbers or with a decimal by a natural number, Euclidean division with two natural numbers (whole quotient and remainder);
- deliberate calculus exact or approximate: organization and processing of a calculus (in one’s head or with help of writing), order of size of a result;
- to use calculators and to command some of their functionalities.

Space and geometry

- locating squares or points in a grid system;
- use of planes and maps;
- geometry connections and properties: alignments, perpendicularity, parallelism, equality of lengths, axial symmetry, midpoint of a segment;
- use of instruments (ruler, set square, pair of compasses) and techniques (folding, tracing, squared paper);
- plane figures (particularly: triangles and remarkable triangles, square, rectangle, rhombus, circle): recognition, reproduction, construction, description, to divide a figure into more simple figures;
- solids (particularly: cube, rectangular parallelepiped): recognition, reproduction, construction, description, patterns;
- enlargement and reduction of plane figures, connecting with proportionality.

Sizes and measurement

- length, mass, volume (capacities): measurement of the sizes (use of instruments, appropriate choice of the unit), estimation (in order of size), legal units of the metric system (meter, gram, liter, their multiples and their submultiples), arithmetic of these measurements expressed with the help of these units;
- length of a polygon;
- areas: comparison of surfaces according to their areas, differentiation between area and perimeter, measurement , measurements of areas with the help of a given unit, common units (cm^2, dm^2, m^2, km^2) and their connections;
- area of a rectangle;
- angles: comparison, reproduction;
- to mark off time and durations: reading time, units of durations (year, month, week, day, minute, second) and their connections;
- calculation of duration spent between two given instants.

ITALIAN LOWER SECONDARY SCHOOLMATHEMATICS STANDARDS

Knowledge	Abilities
The number • The number sets and the properties of the operations. • Multiples and divisors of a number; prime numbers, least common multiple, greatest common divisor. • Quotients, percentage and proportions. • Rational numbers; decimal representation of the rational numbers; comparison between relative rational numbers. • Decimal and periodic numbers; examples of irrational numbers. • The square root as the inverse operation of the second power. • Order of size, approximation, error, aware use of the computational tools. • The formal writing of the properties and the use of the letters as a generalization (from the number to the symbol). • Basic elements of the algebraic calculus. • Simple first grade equations.	• To solve problems and calculate easy expressions with rational numbers. • To decompose a natural number into its prime divisors. • To read and write numbers in decimal base using the polynomial and the scientific representations. • To recognize equivalent fractions; to compare rational numbers and represent them on the number line. • To recognize the different number sets with their formal properties and to operate within them. • To make easy sequences of approximated computations. • To use letters to denote the main properties of operations. • To explore modelizable situations with easy equations; solution of equations in easy cases.
Geometry • Plane figures; significant elements and characteristic properties of triangles and quadrilaterals. • The sum of the angles of a triangle and of a polygon. • Equivalent decomposition of simple polygonal figures. • The theorem of Pythagoras. • Intuitive notion of geometrical transformation: translations, rotations, symmetries. • Similarity. • The length of the circumference and the area of the circle. • The meaning of π and related historical remarks. • The concept of reference system: the Cartesian coordinates, the Cartesian plane. • The solids; computation of the volume of the main solids (cube, parallelepiped, pyramid, cone, cylinder, sphere).	• To know the properties of solid and plane figures; their classification according to different types of properties. • To build up isometric figures with given properties. • To use the transformations to observe, classify and discuss the properties of the figures. • To compute the perimeter and the area of plane figures; to calculate the length of circumferences and the area of circles. • To represent points, segments and figures on the Cartesian plane. • To solve problems by the use of geometrical properties of the figures, also making use of physical models, easy deductions and suitable representations or instruments (ruler, set square, compass and, in case, geometric software). • To compute the surfaces and the volume of the main solid figures. • To visualize 3D objects from a 2D representation and, conversely, to give a plane representation of a solid figure.

Knowledge	Abilities
The measure • The international measure system.	• To express measures by means of the measure international system, by using the powers of 10 and significant figures. • To calculate and express a measure in direct and indirect way. • To recognize proportional measures in different contexts; to scale off. • To evaluate the significance of the figures of the result of a given measure.
Relations • Intuition of the set notion and introduction of the elementary operations between sets. • Some significant relations (to be equal to, to be a multiple of, to be greater than, to be parallel or perpendicular to, …). • Functions: tables and diagrams. • Functions of type y=ax, y=a/x, $y=ax^2$ and their graphs. • Simple models of experiments and of mathematical laws.	• In different contexts, to identify, denote or build up meaningful relations; to recognize analogies and differences. • To use letters to denote in general form simple properties (numerical, geometrical, physical, … properties). • To recognize relations among measures in real contexts. • To use the Cartesian plane, tables and diagrams to represent relations and functions.
Data and estimates • Data collection of continuous quantities: construction of intervals of equal or different amplitude. • Histograms. • Relative or cumulated frequencies and percentages. • Official data sources and their use. • Adequate understanding of the different notions of probability: classic, frequentist and subjective.	• Construction of histograms and their interpretation. • To identify a problem affordable with a statistical survey, to identify the population and the statistical units related to the problem, to prepare a questionnaire, to collect data, to organize them in frequency tables. • To represent by a diagram and to analyse the indexes adequate to the characteristics (the mode if qualitatively incoherent, also the mean if they may be ordered, the arithmetic average if quantitative). • To estimate, in easy contexts, the possible results of an experience and their probabilities. • To recognize wrong diagrams and to correct them, if possible. • To get information from data collections and graphics from different sources. • To use IT tools to organize and represent data. • To compute relative frequencies and percentages and understand their meaning. • To use relative or cumulated frequencies and percentages to compare data collections. • To understand when and how to use the different probability measures.

<table>
<tr><th>Knowledge</th><th>Abilities</th></tr>
<tr><td colspan="2">Rational thinking (to be coordinated to all other studies and educational subjects)
• From the natural to the formal language; to understand and to use suitable words for different contexts; the propositions and the introduction of the logical connections non, et, vel.
• To understand the role of definitions.
• To use different logic processes: induction and generalization, deduction. The meaning and use of examples and counter-examples.
• To conjecture about observations in different contexts.
• To properly motivate enunciations, recognizing statements induced by the observation, guessed and conjectured, discussed and proved.
• To recognize problems, data and goals.
• To depict and schematize in different ways a problem, in order to detect a possible way to solve it.
• To prove the processes chosen and implemented in the problems solutions.
• To express clearly the solution, the necessary steps and their connections.
• To evaluate critically the different strategies to solve a problem.</td></tr>
</table>

Table 10. Italian lower secondary school mathematics standards

ITALIAN INPUT SITUATION
SUMMARY OF MATHEMATICS TAUGHT AT THE PRIMARY LEVEL

Arithmetic

- Natural numbers and their notation in the decimal system.
- Their presentation on a number line.
- Comparison of natural numbers.
- Operations with natural numbers including division by one - and more - digit divisors and powers of natural numbers.
- Solving word problems resulting in one or more operations with natural numbers.
- Decimal numbers, their addition and subtraction.
- Using decimal numbers in real contexts (money, weight, length).
- Fraction as a part of a whole, representation of fractions.

Geometry

- Recognition of 2D-figures (triangle, rectangle, square, quadrilateral, circle).
- Measuring lengths of line segments - units of length: mm, cm, dm, m, km and their conversions.
- Perimeter of a figure.
- Construction of simple regular geometrical figures.
- Symmetries and simple geometrical transformations of geometrical figures.
- Units of area mm^2, cm^2, dm^2, m^2.
- Determination of the area of simple polygons (from the triangle to the regular hexagon) using formulas.
- Solving word problems resulting in simple calculations of perimeters and areas of simple polygons.

The measure

- Measures by means of the measure international system, converting them into each other in real simple cases.
- Determination in simple cases of lengths, areas and volumes.

Data and estimates

- Data collection of quantities.
- Histograms.

Rational thinking

- To use the suitable mathematics words formerly introduced in different contexts.
- To use examples to verify hypotheses and conjectures.
- In a word problem, to be able to detect data and their connections and to plan a solution to solve it.

SLOVAK LOWER SECONDARY SCHOOL MATHEMATICS STANDARDS

Knowledge	Abilities
Natural numbers • Reading and writing; • Representing on the number axis; • Comparing and rounding; • Adding, subtracting, multiplying and dividing (also with the remainder); • Solving the word oriented tasks; • Attributes of divisibility; • Prime number, composed number, factoring; • Divisor, common divisor, the greatest common divisor; • Multiple, the common multiple, the lowest common multiple; • Algorithmization of the calculation of the greatest common divisor and the lowest common multiple.	• Writing in shortened and expanded forms; • Assigning the mirror of a number on the number axis; • Solving 11 types of simple word oriented tasks; • Solving composed word oriented tasks with at most three computing operations; • Dividing with remainder, making the check of the solution; • Calculating a multiple of a given number, common multiples of two numbers, the lowest common multiple of two numbers; • Using criteria of divisibility by numbers 2, 3, 4, 5, 10; • Calculating divisors of a given number, common divisors of two numbers, greatest common divisor of two numbers.
Decimal numbers • Concept of a decimal number, order of decimal numbers; • Comparing and rounding; • Adding and subtracting; • Multiplying and dividing; • Word oriented tasks; • Tasks on continual proportion and arithmetic average.	• Reading and writing decimal numbers; • Representing decimal numbers on the number axis; • Comparing and rounding off decimal numbers; • Simple adding and subtracting of decimal numbers by rote; • Multiplying a decimal number by at most a three-digit number; • Dividing: a smaller natural number by a bigger natural number (by two-digit number at most), a decimal number by a natural number, a natural number by a decimal number, a decimal number by a decimal number; • Solving 11 types of simple word oriented tasks within the range of decimal numbers.
Integer numbers • Positive and negative numbers; • Opposite numbers; • Number axis, interpretation of an absolute value of a number; • Organizing integer numbers; • Negative decimal numbers; • Operations with integer numbers; • Problem solving.	• Recognizing integer numbers; • Reading and writing down integer numbers • Giving pairs of opposite integer numbers and knowing their basic characteristics; • Comparing integer numbers; • Adding and subtracting integer numbers readily, multiplying and dividing by rote and in writing; • Solving word oriented tasks with integer numbers.

Knowledge	Abilities
Fractions, rational numbers • Concept of a fraction; • Equation of fractions; • Fraction abbreviation and extension; • Representing fractions as decimal numbers; • Rational numbers; • Organizing fractions according to their value; • Mathematical operations with fractions.	• Understanding a fraction correctly; • Reading and writing down a fraction; • Expressing a part of a whole in form of fraction; • Comparing fractions and marking the result of comparing with predicate symbols <, >, = ; • Expanding and abbreviating fractions; • Converting a fraction to its basic form; • Writing down fractions in a form of decimal numbers and vice versa; • Understanding the notion of rational numbers correctly; • Representing a rational number on the number axis; • Comparing rational numbers using <, >, = ; • Understanding the notion of a mixed number correctly; • Writing down a mixed number in the form of a fraction and vice versa; • Adding and subtracting fractions and mixed numbers; • Calculating fractional part of a whole; • Multiplying fraction by another fraction; • Dividing integer number by a fraction and vice versa; • Dividing a fraction by another fraction; • Solving word oriented tasks based on operations with rational numbers.
Percents • Concept of a percent; ground; • Strength of percentage; • Part appertaining to the strength of percentage; • Bar and circular charts; • Solving word oriented tasks on percentage.	• Determining 1 percent; • Distinguishing, naming and calculating the ground, strength of percentage and value appertaining to the strength of percentage; • Estimating and determining the strength of percentage by means of circular or bar charts; • Calculating the interest, the principal (or the capital) and the interest rate by yielding a simple interest; • Solving word oriented tasks on percentage from the area of finance.
Ratio; Direct ratio and indirect ratio • Concept of a ratio; • Inverted ratio; • Sequential ratio; • Direct and indirect ratio; • Graph of direct and indirect ratio; • Simple and compound rule of proportion; • Scale of plans and maps; • Solution of tasks.	• Correct understanding of the terms ratio, inverted ratio, sequential ratio and knowing how to use them in a solution of the tasks; • Dividing, multiplying and diminishing a given number in a given ratio; • Marking points in a rectangular system of coordinates in the plain; • Understanding the right meaning of the outright and indirect proportions and knowing how to represent it in the coordinate system in the plane; • Knowing how to read from the graphs of the outright and indirect proportions;

Knowledge	Abilities
Ratio; Direct ratio and indirect ratio (cont.)	• Solving tasks concerning the outright and indirect proportions; • Solving exercises using a to-scale map and plans.
Powers and roots • Square and cube powers; • Square and cube roots; • Powers with natural exponent; • Operations with powers, with natural exponent, to the 0th power; • Writing down numbers of type a.10n, where 1≤ a < 10.	• Writing down the products of the same factors into the shape of a power and conversely; • Reading the natural power of an arbitrary rational number and designating individual numbers in powers; • Reading square and cube roots of arbitrary positive rational numbers and naming particular numbers in powers/roots; • Calculating the value of the square and cube roots by rote and on a calculator; • Calculating the square of a product; • Calculating the product of square roots of non-negative numbers; • Writing numbers in the form of $a \cdot 10^n$, where $1 \leq a < 10$; • Adding and subtracting powers with the same base and the same natural exponents; • Multiplying and dividing powers with the same base; • Raising products and quotients to a given power.
Algebra • Numerical expressions; • Expressions with variables; • Arithmetic functions with numerical expressions; • Arithmetic functions with whole expressions; • Fractional expressions; • Arithmetic functions with fractional expressions.	• Understanding letters in the meaning of numbers; • Correctly writing algebraic expressions; • Adding, subtracting and multiplying algebraic expressions; • Modifying the algebraic expressions with the help of formulas $(a + b)^2$, $a^2 - b^2$; • Using fractional expressions and giving conditions for them to have a meaning; • Shortening and expanding of fractional expressions; • Adding and subtracting fractional expressions; • Dividing and multiplying fractional expressions by whole and fractional expressions; • Expressing the unknown from the equation.
Linear equations and their systems • Equality of two expressions; • Solving a linear equation with one unknown using equivalent operations; • Solving an equation with unknown in denominator; • Solving systems of two linear equations with two unknowns; • Expressing the unknown from the equation.	• Deciding if two numerical and algebraic expressions are equal; • Equivalent operations with an equation and their use in solving a linear equation with one unknown • Solving a linear equation with unknown in the denominator; • Making the check for the solution of an equation; • Solving the system of two linear equations with two unknowns; • Solving word oriented tasks leading to a linear equation with one unknown or to a system of two linear equations with two unknowns.

Knowledge	Abilities
Linear inequalities • Concept of a solution of a linear inequality; • Step-by-step substitution; • Equivalent operations with a linear inequality.	• Solving a linear inequality; • Multiplying a linear inequality with a negative number; • Solving the word oriented tasks leading to a linear inequality.
Functions; Linear function • Concept of a function; • Dependent and independent variable; • Domain and codomain of a function; • Linear function, graph of a linear function; • Properties of a linear function.	• Recognizing a linear functional dependence of two variables; • Constructing graphs of linear functions using a table for values of dependent and independent variables

Table 11. Slovak lower secondary school mathematics standards

SLOVAK INPUT SITUATION
SUMMARY OF MATHEMATICS TAUGHT AT THE PRIMARY LEVEL

Numbers

Natural numbers. Quantitative aspects of numbers. Notation of natural numbers in the decimal system. Representation on a number line. Sequence of natural numbers. Comparison of natural numbers. Reading and writing of big numbers. Rounding of natural numbers. First acquaintance with decimal numbers and with their use in concrete models (length, weight). Fraction as a part of a whole.

Operations

Addition and subtraction of natural numbers. Multiplication of natural numbers. Algorithm of division of a natural number by one digit divisor.

Elements of algebra

Variable. Equation. Inequality. Solving of inequalities of type $x < a$, $x > b$.

Solving word problems

Simple word problems leading to the use of addition or subtraction or multiplication or division. Compound word problems leading to the use of expressions of types $a + b + c$, $a - b - c$, $a - (b + c)$, $(a + b) - c$, $a + (a + b)$, $a + (a - b)$, $a + a \cdot b$, $a + a:b$, $a \cdot b + c$, $a \cdot b + c \cdot d$.

Geometry

Recognition of 2D and 3D figures: triangle, square, rectangle, circle, sphere, cube, cylinder. Simple lines and curves. Open curves, closed curves. Point, line segment, line. Units of length (mm, cm, dm, m, km), length of a line segment. Construction of line segments of given lengths. Conversions of units of length. Construction of a circle. Construction of lines parallel to a given line. Square grid. Determination of the area of a rectangle using the square grid. Calculation of the area of a rectangle. Units of area (mm^2, cm^2, dm^2, m^2, ha). Triangle, quadrilateral - notation of vertices and construction. Construction of a rectangle and a square. Construction of perpendiculars. Addition and subtraction of lengths of line segments. Perimeter of a triangle, rectangle, square. Solving word problems leading to simple calculations of perimeters and areas of rectangles and squares.

3. COMPARISON OF TEACHER TRAINING SYSTEMS

The variety of European teacher training systems is clearly shown by the tables below, where only the five project's partner countries are compared. Due to different educational traditions, cultures and societies, this variety makes it hard to come up with a reasonably fair proposal for a single European teacher training system.

The tables below are therefore meant to be a tool for reflection, to be used by educators and policy makers at both national and European level. Nevertheless, as regards lower secondary school teacher training in mathematics, these tables and, above all, the detailed description of the national educational teacher training systems allowed the Project Team to point out communalities and differences (see Chapter 4), and to get a general idea of both advantages and disadvantages of the examined systems. This sort of outcomes, together with ideas by stakeholders in the field of education will form the basis for a wider debate and possible future suggestions for changes at European level, in view of making teachers' mobility throughout Europe really possible, at both administrative and professional level.

TEACHER TRAINING SYSTEMS								
	Initial Degree	**Age at entrance**	**Admission**	**Final Qualification**		**Years**	**Final Proof**	**Age at exit**
				Teacher at	***Teacher of***			
CZ	UpSS	**19**	Budget	LoSS *or* LoSS and UpSS	Maths + 1 other subject	5	Exam	**24**
DK	UpSS	**19**	Open	Pr and LoSS	Maths + 3 other subjects	4	Exam	**23**
FR	Un (3 years)	**21**	Programmed	LoSS and UpSS	Maths	2	Contest	**23**
IT	Un (4-5 years)	**23**	Programmed	LoSS	Maths & Science	2	Exam	**25**
SK	UpSS	**19**	Budget	LoSS *or* LoSS and UpSS	Maths + 1 other subject	5	Exam	**24**

Table 12. Teacher training systems comparison

***Key*:**

Pr Primary School.
LoSS Lower Secondary School
UpSS Upper Secondary School
Un University
+ Students are trained to teach one or more additional subjects besides MATHS
& Students are trained to teach just one subject: Maths and Sciences
Open no restrictions
Budget number of admittances determined according to the Universities budget
Programmed number of admittances determined according to the foreseen future needs

TEACHER TRAINING STRUCTURES				
	Years of Education	**Year**	**Subject area**	**Hours**
CZ	5	**1-2½** **2½-3** **4-5**	Basic Mathematics + Didactics/Pedagogy/Psychology Pedagogy/Psychology + Passive Practice Basic Mathematics + Didactics of Mathematics, Active Practice[7] (4 weeks)	1100 (BasMat+DidMat) + 560 (Ped/Psy + PasPra[8])
DK	4		Basic Maths + Didactics of Maths + Didactics/Pedagogy/Psychology + Practice	(220→395) BasMat + DidMat + Did/Ped/Psy + (24 weeks) Pra
FR	**2** ***- - - or - - -*** 4/5 Un + Public Contest (*Agrégation*) **+1**	**1** **2** **1**	Basic Mathematics – Public Contest (*CAPES*) Didactics/Pedagogy/Psychology + Practice ***- - - or - - -*** Didactics/Pedagogy/Psychology + Practice (at IUFM)	**1** 500 BasMat **2** 216 Did/Ped/Psy + 216 Pra ***- - - or - - -*** **1** 216 Did/Ped/Psy + 216 Pra
IT	2	**1** **2**	Didactics/Pedagogy/Psychology + Basic Mathematics + Sciences + Passive Practice – Exams Didactics/Pedagogy/Psychology + Basic Mathematics + Sciences + Active Practice – Final Report – Exams	**1** 100 Did/Ped/Psy + 90 BasMat + 140 Sci + 170 PasPra **2** 100 Did/Ped/Psy + 60 BasMat + 110 Sci + 130 ActPra + 100 FinRep
SK	5	**1-3** **4-5**	Didactics/Pedagogy/Psychology + Basic Mathematics + Subject 2 + Practice (only 6th sem, 2h/week, half passive + half Active) Basic Mathematics + Other Subject 1+ Didactics of Mathematics + Other Subject2 + Active Practice (7th + 8th sem, 4h/week) + Continuous Practice (10th sem., 4 weeks)	481 Did/Ped/Psy + 988 BasMat + 988 Sub2 (this includes Didactics of both BasMat and Sub2) + 130 Pra (65 in each of BasMat and Sub2) + (4 weeks) ConPra for BasMat and Sub2

Table 13. Teacher training structures comparison

[7] 2 weeks at Lower Secondary School, 2 weeks at Upper Secondary School; approximately the same number of hours for the second subject.

[8] Pedagogy/Psycology + Passsive Training + a set of courses (e.g. philosophy, sociology, languages, economy, political sciences, informatics etc.).

CZECH LOSSTT-IN-MATH SYSTEM

Mathematics in the educational system

The role of mathematical education is very important. Mathematics should be considered as one part of human culture. Mathematics develops independent thinking of pupils by problem solving and logical thinking through its deductive build-up. With introducing concepts like variable, expression, equation, inequality, function etc., mathematics develops pupils' abstract thinking and contributes to their general intellectual development. Geometry encourages pupils to be precise through geometric constructions and helps them to develop logical thinking by proving correctness of constructions and by verifying correct number of solutions to a given problem. Its role is irreplaceable. It also helps pupils to develop spatial imagination by solving problems from solid geometry and functional thinking by solving construction problems based on congruent mappings and homothety. Mathematics also brings pupils to understanding of the concept of infinity both in the context of a sequence and its limit and in the context of geometry. In topics dedicated to combinatorial analysis, probability and statistics, mathematics contributes to the development of combinatorial and probabilistic thinking of pupils and encourages them to solve problems taken from life (transport, management, theory of prognoses). An important role is played by such problems that are used by teachers for explaining strategy of solution and process of converting a real problem into a mathematical one and vice versa and for leading pupils to a correct evaluation of results and their number and possibilities of using computer technology. Besides, mathematics is an important tool for other subjects.

Teacher training in mathematics

The future primary school teachers (from the first till the fifth grade) take a four-year study. They get acquainted with all fields of study taught at this stage of schools. Their knowledge of mathematics involves elementary arithmetic, set theory, relations, mappings, algebraic structures, mathematical logic, essentials of geometry etc. Besides mastering these mathematical theories they are instructed about methods and efficient procedures of teaching.

The future lower and upper secondary school teachers (from the sixth till the twelfth grade) take a five-year study. Mathematics is combined with other subject (e.g. physics, descriptive geometry, computer science, chemistry, biology, geography, arts, and foreign languages). Faculties gained certain autonomy after 1989, which influenced the non-unified system of teacher training. Most faculties have separated the lower and upper secondary level, some of them offer either lower secondary level or lower and upper secondary level.

The traditional way of training prospective teachers of mathematics emphasized scientific knowledge. Nowadays, facing the necessity of preparing teachers for a new flexible school system, we concentrate much more on the didactical aspects of teacher training. The starting point is to state which parts of mathematics are necessary for future mathematics teachers.

To maintain the balance between humanization of education and the growth of demands for mathematical literacy in both children and adults, the graduate secondary school teacher should

- have good and systematic knowledge of mathematical disciplines taught at basic or secondary school, esp. of algebra, geometry and calculus; basic knowledge of psychology, mathematical education and pedagogy; the extent, depth and structure of knowledge sufficient to encourage and fix self-confidence as an expert and teacher; understanding of mathematics as the integrator suitable and necessary for modeling and solving problems from life;
- know how to continue and apply knowledge to solve standard and more difficult mathematical problems, be able to find the possibility of modeling mathematically a real problem, to choose the appropriate mathematical method for solving it and to interpret correctly the solution;
- have basic knowledge of function and use of computers, of algorithmisation of problems and programming and know how to use them in teaching;
- have basic knowledge of logical structure of mathematics and be able to use it;
- be acquainted with the structure of school mathematics, together with connections to previous and following school levels;
- have necessary knowledge of numbers, have functional thinking, developed geometrical plane and space imagination and have a good command over probabilistic and statistical methods;
- be able to communicate in the oral and written form with experts as well as with pupils and to express mathematical ideas and procedures clearly and without mistakes in language;
- realize the liaisons between the structure of mathematics as science and as teaching subject; be able to transform his scientific knowledge into his teaching in the way corresponding with his pupils' level;
- have a basic knowledge of organization and management of the teaching procedure, diagnostics and ways of evaluation;
- be ready to use alternative ways of teaching and differentiate between different groups of pupils;
- have the basic survey of philosophy and history of mathematics and mathematical education;
- understand the necessity to continue his studies in mathematics and to improve his pedagogical competence and be able to do it; follow the appropriate literature and foreign sources.

Not all faculties in the Czech Republic have moved towards this new system of teacher training, some of them stress only the pure mathematical content and do not care for the didactical aspects. Anyway, we can see the massive tendency towards creating didactical schools.

Module of the subject base - mathematics, 1st cycle (Year 1 - 3):

Elementary mathematics (52 hours)

Introduction to calculus (39 hours),
Calculus of functions of 1 variable (104 hours),
Sequences and series (39 hours),
Functions of several variables (39 hours)

Introduction to algebra (39 hours),
Linear algebra (52 hours),
Polynomial algebra (39 hours),
Algebraic structures (39 hours)

Elementary geometry (78 hours),
Conics (39 hours),
Analytic geometry (65 hours)

Computers and informatics (52 hours)

Problem solving (26 hours)

Compulsory modules of the 2nd cycle (Year 4 - 5):

Module of Didactics of mathematics

Didactics of mathematics (104 hours),
Problem solving (26 hours),
History of mathematics (26 hours),
School practice (4 weeks)

Module of mathematics

Set theory (39 hours),
Probability and Statistics (52 hours),
Finite mathematics or Numerical methods (39 hours)

Optional modules of the 2nd cycle (Year 4 - 5):

(8 courses to be chosen from the offer according to the student's preferences)

Module of mathematics

Differential equations, Functional analysis, Application of calculus, Number theory, Symmetries in algebra, Equations and their systems, Curves and surfaces, Geometry of Gauss plane, non-Euclidean geometries, programming, Mathematical software, Modern mathematics.

Module of Didactics of mathematics

Research in Didactics of mathematics, Diploma thesis seminar, Problem solving.

Special course offered in co-operation with the Department of English language and literature: CLIL – teaching mathematics in English.

DANISH LOSSTT-IN-MATH SYSTEM

Aims of teacher education

The education is based on a concurrent rather than a consecutive model. We aim at integration between subjects and theoretical, professional and practical teacher preparation.

According to the executive order of June 19, 1998, the objectives of teacher education are:

- that the student teachers obtain professional insight into the educational theory, subject areas, and educational practices that are necessary for their careers as teachers in the Danish "Folkeskole" and for managing other kinds of tuition and instruction at similar levels,
- that the student teachers learn to co-operate, plan, execute and assess teaching, employing their theoretical and practical knowledge,
- that the personality of the student teacher is formed and developed through independent work with the subject matter, through co-operation and through joint responsibility for his/her own education, and
- that the student teachers' involvement in, enthusiasm and joy of working with children and adults in the process of schooling are furthered.

Structure of the education

Colleges of Education are the only institutions authorized to educate teachers for the Danish "Folkeskole", a municipal school combining primary and lower secondary education and run on a comprehensive basis for children between the ages of 7 and 17.

The education takes four years. Each study year is divided into two terms with an annual workload of 1,680 hours. The education consists of seminars, lectures, study weeks, teaching practice periods and project periods including tutorials. The education comprises the following subjects and subject areas:

1. Religious studies and philosophy
2. The educational subject areas:
   - philosophy of education (pedagogy)
   - psychology
   - general didactics
   - educational sociology (school and society)
3. Teaching practice
4. Four specialized subjects:

At Skårup we offer the following specialized subjects:

Humanities:

Danish
English
German
History
Religious studies
Social science

Natural sciences:

Biology
Geography
Mathematics
Physics/Chemistry
Science and Technology

Practical/aesthetic subjects:

Art
Domestic science
Music
Sports
Woodwork

According to Danish "Grundtvigian" tradition, the exchange of ideas is a vital element in all courses; but this on-going dialogue between tutors and students is interrupted by group projects or individual studies.

At present, teacher training is offered at 18 colleges distributed all over Denmark. The colleges are administered by the Ministry of Education and are under ministry supervision both with regard to financing and to the education offered. The colleges of education are the only institutions which are authorised to provide the 4-year programme, which qualifies for teaching posts in the Danish Folkeskole.

Duration and Structure of the Programme

The teacher training programme is of 4 years' duration.

The *programme* includes teaching practice at a school for a total of 24 weeks, the organization of which is decided by the individual institution.

The programme includes the following subjects:

Common core subjects

Theory of education, psychology, general didactics, school and society (70%),
religious studies and philosophy (20%),
teaching practice (60%),
thesis (15%)

Main subjects

Danish or mathematics (70%) and three further main subjects (each 55%) must be chosen by the student.

The four main subjects must be chosen so that at least two of the following three areas are represented: humanities, natural sciences and practical-aesthetic subjects (the percentages indicate the proportion of a student's full-time workload devoted to these subjects during one year).

Admission and Student Population

The admission requirement for the teacher training programme is one of the following leaving examinations at upper secondary level: the upper secondary school leaving examination (the "Studentereksamen"), the higher preparatory examination (the "HF-eksamen"), the higher commercial examination (the "HHX") or the higher technical examination (the "HTX").

Due to the fact that there are normally more applicants than available study places, it has been necessary to introduce restricted admission to the teacher training programmes.

25% of the study places are awarded on the basis of the leaving examination at upper secondary level. As for the other 75%, it is in principle up to the colleges themselves to decide, but the colleges have agreed on common rules which are administered locally:

- the result of the leaving examination ("the higher, the better"),
- practical work experience,
- folk high school attendance,
- stay abroad,
- experience with children (youth center, boy/girl scout movement etc.).

1/3 of the students are men, 2/3 are women.

About 55% of the students complete the programme at the end of the officially stipulated time of study, and 75% in total complete the programme.

Legislation

Consolidation Act no 608 of 10 July 1997 on the Training of Teachers for the Danish Folkeskole only indicates the general lines of the programme: the duration, subjects, rules regarding the institutions and their management etc.

Ministerial Order no 382 of 19 June 1998 regulates the scope of the subjects, the principal lines of the content of the subjects, and general rules for the assessment of students.

The more detailed regulations are laid down in local curricula drawn up by the institutions.

Qualifications of the Teachers at the Colleges of Education:

The teachers at the colleges of education may have three different kinds of qualifications:

- A university degree (Master's level) in humanities or in natural sciences subjects.
- An academic degree (Master's level) from the Royal Danish School of Educational Studies.
- A Folkeskole teacher qualification supplemented with in-service training (mostly in non-academic subjects such as textile design, wood/metalwork etc.).

THE COMPETENCE OF DANISH FOLKESKOLE TEACHERS

A Danish teacher's certificate obtained at a college of education is based on a leaving examination at upper secondary level, and the programme is of 4 years' duration.

The colleges of education are the only institutions which are authorized to provide the 4-year programme, which qualifies for teaching posts in the Danish Folkeskole.

Danish colleges of education offer a programme which in scope and level can be said to correspond to certain types of university programmes in the English-speaking world. The qualification which students obtain on completion of the programme can most aptly be compared to a UK/US Bachelor's degree. All the colleges are under ministry supervision. The quality of the examinations is controlled by external examiners appointed by the ministry.

In theory, a teacher's certificate qualifies the graduate teacher to teach all subjects to all forms (1st-10th forms), but in fact the teacher is generally considered competent to teach the 1st to 10th forms in the four main subjects taken.

In practice, the authorities responsible for the appointment of teachers (i.e. the municipalities, including the school board and the head-teacher of the individual school) take the final decision with regard to the question of competence.

SUMMARY OF MATHEMATICS TEACHER TRAINING	
Admission:	Upper secondary leaving examination (studentereksamen) or equivalent
Duration of Teachers' education	Teacher education takes four years Each study year with an annual workload of 1680 hours. The four years gives 240 ECTS.
students	After ending teachers college they have permission to teach children in the age of 7 to 17 (Folkeskolen) in four subjects.
Mathematics	It is compulsory to choose between Danish or mathematics (subject 1). Subject 1 gives 42.0 ECTS.
Duration of math.	Normally it takes three to four years
Number of lessons in Math.	220–395
Number of weekly lessons in math.	4 in average
Numbers of students	
Numbers of teacher colleges	18 in Denmark
Language	Danish
Examination	One written examination. Common for the whole country. An oral examination in which the student are examined in mathematic, didactic, and practical- pedagogical aspects.
Literature	There is written one system for teachers education in mathematic. A lot of teachers are using only there own notes supplied with copies from elsewhere. Many teachers are using a Norwegian book of didactic.
The teacher	Are educated at the university or are at first educated at a teachers college and afterwards at a pedagogical university.
Size of class	Normally we are teaching in a class with 20 - 30 students. Mostly we have no specialization. We teach in all aspects of mathematics.
Practice	24 weeks of teaching practice.
Our role in practice	We give guidance before starting practice. Are visiting the students while they are in practice and discuss it afterwards.
Examples of the working	We are trying to vary the forms: • Normal classrooms activities • Experimental works, both in groups and single work • Working with IT in the classroom • Working with other activities like story line and learning tables with the hole body etc.

Table 14. Mathematics teacher training in Denmark

Objectives/Aims

To acquire:

1. knowledge in subjects, central in mathematics and in math teaching in school;
2. knowledge in using argumentation, experimentation, investigation, organizing and generalizing in mathematics;
3. knowledge in development of children's concept formation in mathematics.

Central Knowledge and Proficiency areas

Numbers

- Numbers, scale, cardinal numbers.
- The basic arithmetical operations and their hierarchy.
- Reasons for selected general rules in arithmetic.
- Formulas and generalizing.
- Representation and working up data.
- Fundamental concept of statistics.
- Probability and testing of hypotheses.
- Models, growth and simulations.

Geometry

- Plane geometrical fundamental concepts and demonstrations.
- Analytical geometry.
- Spatial geometry and its representation in the plane.

Didactics

- The reasons, contents and historical development of the school subject.
- Determination of objectives, choice of content and planning of a delimited course of teaching.
- Choice and preparation of teaching materials.
- Use of information- and communications technology.
- Differentiation in teaching . Evaluation.
- Views of learning providing the background for different kinds of mathematics teaching.
- The language of mathematic as a means of communication.

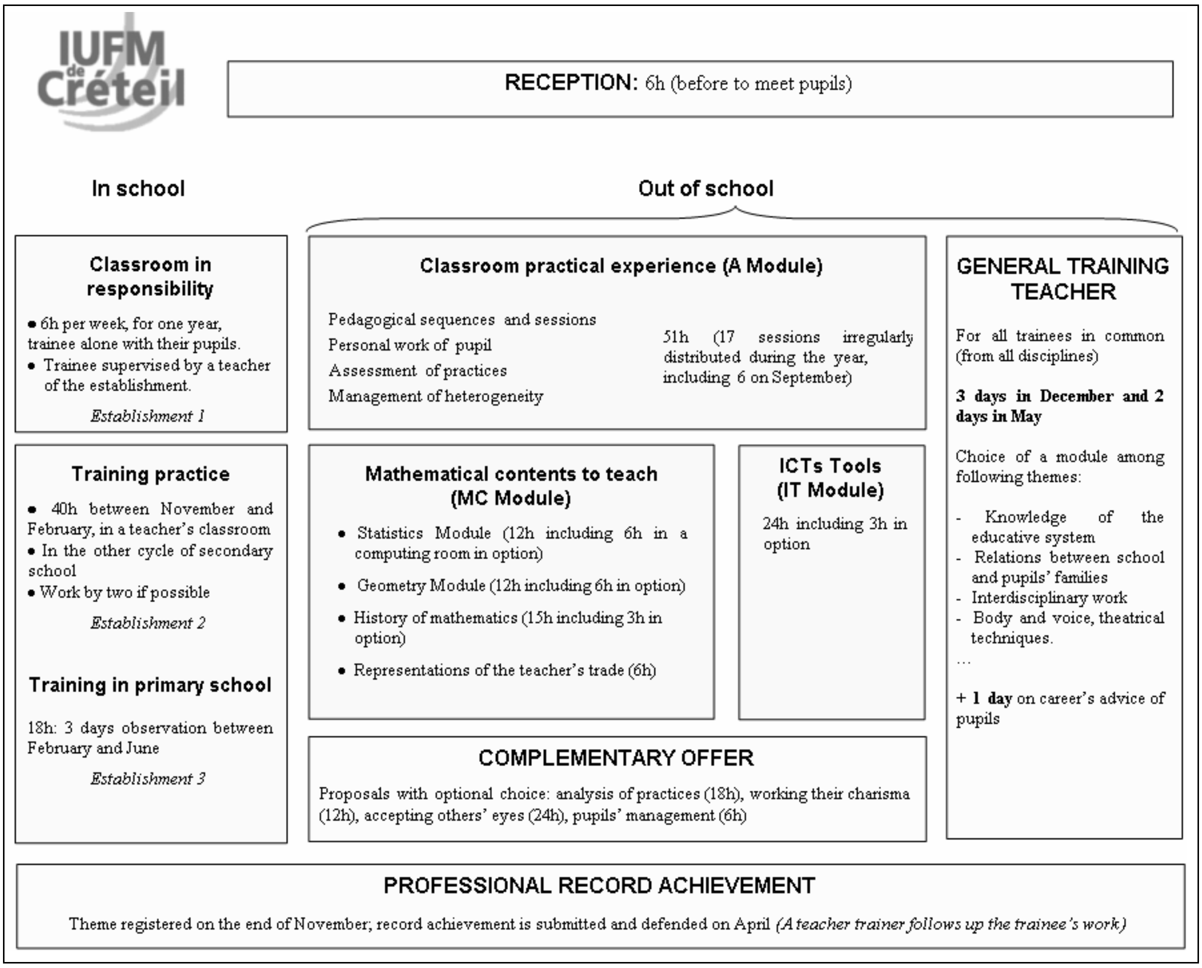
FRENCH LOSSTT-IN-MATH SYSTEM

Training is unfolding during two years:

1. First year is focused on the preparation of a competitive examination. The examination evaluates mathematical knowledge of candidates.
2. The second year is devoted to obtain and develop professional skills.

The organisation of this second year is set out in the table below. It joins together individual training in school and training by groups out of school.

1. Training in school (around 250h)
   - Each trainee is responsible for at least one class in one of the two cycles of French secondary education system (lower or upper) during the whole year. He is supervised by a mathematic teacher of the same school.
   - He has also a training period (40h), in another teacher's classroom in the other cycle, and also an observation period in primary school.
2. Training out of school (around 120h)
   - In mathematics: classrooms practices, mastering contents to teach, ICTs, professional record achievement.
   - In general subjects connected to their teaching.



Picture 6. Mathematics teacher training in France

ITALIAN LOSSTT-IN-MATH SYSTEM

SSIS - Specialization school for secondary school teachers

General information:

- post-graduate University biennial courses
- ruled at national level
- organized at regional level, within the frame of inter-University agreements
- regional administrative structures and educational models
- parallel didactical activities in each University in the Region
- educational activities organized in different disciplinary sections
- admission by public contest at regional level (two written proofs: discipline knowledge and teaching abilities)
- educational requirement: *laurea* (University degree: 4/5 years)
- interim exams at the end of each of the four semesters
- final State exam, entitling to be a teacher of a set of subjects related to the chosen section.

Specific information:

- in Tuscany: agreement among the Universities of Pisa (administrative management), Firenze and Siena
- one thousand hours of classes to be attended, as for the following table:

Type of activity	First year	Second year
General pedagogy	100 hours	100 hours
Discipline (3/5 *theory* - 2/5 *didactical laboratory*)	230 hours	170 hours
School Practice	170 hours	130 hours
Final report on the school practice		100 hours

Table 15. Teacher training in Tuscany

- disciplinary section for future Mathematics teachers in lower secondary schools: *Natural Sciences*
- educational requirement: *laurea* in any scientific discipline
- disciplinary instruction given through four semester courses of *Mathematical Sciences* and four semester courses of *Experimental Sciences*, according to the following timetable:

***Discipline* (3/5 theory - 2/5 didactical laboratory)**	**First year**	**Second year**
Mathematical Sciences	90 hours	60 hours
Experimental Sciences	140 hours	110 hours
Total	**230 hours**	**170 hours**

Table 16. Teacher training in mathematics and science in Tuscany

- discipline entitled to teach: Mathematical Sciences and Natural, Physical and Experimental Sciences

ITALIAN LOSSTT IN MATHEMATICAL SCIENCES

The curricula in Tuscany region

SSIS of Tuscany at Florence

First Year - 85 hours

- Arithmetic - 35 hours
- History of mathematics (*The number and its operations*) - 11 hours
- Statistics - 8 hours
- Logics - 8 hours
- Didactics of mathematics - 15 hours
- Computer laboratory - 8 hours

Second Year - 65 hours

- Geometry - 35 hours
- History of mathematics (*from* The geometry *to* The geometries)- 11 hours
- Didactics of mathematics - 6 hours
- Theoretical/practical proof - 6 hours
- Computer laboratory - 7 hours

SSIS of Tuscany at Pisa

First Year - 90 hours (54 hrs. of theory and 36 hrs of didactical and IT laboratory)

- Plane geometry - 25 hours
- Number systems - 20 hours
- Algebra, variables, functions - 20 hours
- Statistics - 10 hours
- Mathematics and physics laboratory - 15 hours

Second Year - 60 hours (36 hrs of theory and 24 hrs of didactical and IT laboratory)

- Space geometry - 30 hours
- Probability - 15 hours
- Intercultural mathematics - 5 hours
- Difficulties in mathematics - 10 hours

SSIS of Tuscany at Siena

150 hours in two years

- Didactics of the mathematics - 35 hours
- Problems and problem solving - 15 hours
- Logics and set theory - 20 hours
- Arithmetic and introduction to algebra - 30 hours
- Geometry - 35 hours
- Probability and statistics - 15 hours

SLOVAK LOSSTT-IN-MATH SYSTEM

At the national level

The future basic school teachers at the primary level (1st to 4th grades) take a four-year study during which they get acquainted with all fields of the subject matter taught at the primary level of basic schools. As far as the Mathematics content of the studies is concerned, they take courses in elementary set theory and propositional logic, arithmetic, algebraic structures, essentials of geometry, basics on computers, etc. However, in Mathematics as well as in other subjects, the emphasis is on didactical aspects of their future teaching, that is, an essential part of the courses is devoted to theory and practice regarding methods and efficient procedures of teaching.

The future basic school teachers at the lower secondary level (5st to 9th grades) take a five-year study aimed at their preparation for teaching two subjects. Mathematics is combined with other subject (e.g. physics, chemistry, biology, geography, computer science, arts, and foreign languages). Pedagogic Faculties in Bratislava and Banska Bystrica prepare future teachers only for the lower secondary level while Faculties of Mathematics, Physics and Informatics in Bratislava and of Natural sciences in Kosice prepare teachers for both lower and upper secondary level (5st to 12th grades).

At the local level

Pedagogic Faculty of Matej Bel University in Banska Bystrica

In the last years LOSSTT in Mathematics at Pedagogic Faculty in Banska Bystrica was combined with LOSSTT in the following subjects: the Slovak language or Arts or Music or Physical education. Below are compulsory and optional courses and so called ‘compulsory optional modules’ of the Mathematics curriculum valid in this form since 1996:

Compulsory courses of the 1st cycle (Years 1 - 2):

Introduction to Mathematics (39 hours);

Geometry (26 hours),
Mappings (39 hours);

Algebra 1 - Algebraic structures (39 hours),
Algebra 2 - Polynomial algebra (26 hours),
Algebra 3 - Linear algebra (26 hours);

Mathematical analysis 1 (52 hours),
Mathematical analysis 2 (52 hours).

Compulsory courses of the 2nd cycle (Years 3 - 5):

Measure 1 (26 hours),
Drawing geometry (26 hours),
Analytic geometry (26 hours);

Arithmetic 1 (13 hours);

Didactics of mathematics A (26 hours),
Didactics of mathematics B (26 hours);

Computers and programming 1 (13 hours);

Discrete mathematics (26 hours);

Probability and statistics (26 hours);

Application of mathematics in science (13 hours).

Compulsory optional modules:

A - ***Module of geometry***
Nonlinear mappings (26 hours), Measure 2 (26 hours), Conics (26 hours), Drawing tasks (26 hours).

B - ***Module of algebra***
Polynomial algebra (13 hours), Linear algebra (26 hours), Arithmetic 2 (26 hours), Algebraic structures and lattices (13 hours); Mathematical Olympiad (13 hours).

C - ***Module of Didactics of mathematics***
History of mathematics (13 hours), Methods of solving mathematics problems 1 (26 hours), Arrangement of mathematics curriculum (13 hours), Concepts in mathematics (26 hours), Workshops 1(26 hours), Statistics in school practice (26 hours).

A - student's option has to be at least 3 courses;
B - student's option has to be at least 3 courses;
C - student's option has to be at least 5 courses.

Optional modules:

Elementary mathematics (26 hours), Personal computers (26 hours), Internet (26 hours), Outline and draw (13 hours), Romantic mathematics (26 hours), Computers and programming 2 (13 hours), New course in Didactics of mathematics (13 hours), Methods of solving of mathematics problems 2 (26 hours), Workshops 2 (26 hours), Diploma thesis seminar (13 hours).

Trajectory of Teacher training at Pedagogic faculty of M Bel University LOSSTT program in Mathematics with 2nd specialization

Magister study 5 years	Type of course	credits / hours	Total		1st cycle (Year 1-2)		2nd cycle (Year 3-5)	
			credits	hours	credits	hours	credits	hours
Pedagogical-Psychological and Basic module	C		34	26	28	21	6	5
	C/O	50/37	10	7	4	3	6	4
	O		6	4	2	2	4	2
1st specialization (Mathematics)	C		59	40	31	21	28	19
	C/O	110/76	31	20	7	5	24	15
	O		20	16	5	5	15	11
2nd specialization (e.g. Music)	C		59	40	31	21	28	19
	C/O	110/76	31	20	7	5	24	15
	O		20	16	5	5	15	11
Diploma thesis	C	10	10				10	
Pedagogical Practice	C	20	20	10			20	
TOTAL		**300**	**300**	**189**	**120**	**88**	**180**	**101**

Table 17. Teacher training at Matej Bel University

Key: C compulsory, O optional

Pedagogical practice in LOSSTT

In each of the two specializations (Math+2nd Spec) the following is prescribed:

Passive practice	the 6th semester: 13 hours (1 hour per week)	1 credit
Active practice	the 7th semester: 26 hours (2 hours per week)	2 credits
	the 8th semester: 26 hours (2 hours per week)	2 credits
Continuous practice	the 10th semester: 4 weeks in a block of full teaching in school	5 credits

Table 18. Pedagogical practice at Matej Bel University

So out of total 300 credits for the LOSSTT, 10 credits are assigned for pedagogical practice in each of the two specializations (e.g. Math + Slovak language).

Diploma thesis in LOSSTT

- 10 credits
- the number of hours is not specified - it is expected that students will devote at least the last 4 semesters to an intensive work (out of their other duties) to writing their diploma thesis (in one of the two specializations).

Number of students to be accepted for LOSSTT

- no numbers are prescribed at national level (from the Ministry or so), they are determined by each Faculty according to its capacities (personal, financial,…)
- so in reality the numbers of students for LOSSTT at the Pedagogic Faculty are suggested for each academic year by the Dean, subsequently are approved by the Senate of the Pedagogic Faculty and then finally formally approved by the signature of the Rector of the university
- in the last years we in reality accepted all students interested in LOSSTT
- there is some subsidy for each accepted student by the Ministry of Education, but this is not so relevant to become the main issue in `the battle' for students among universities (there are 23 universities in Slovakia which means there are just too many for such a small country and the `survival' fight between them is expected to be tough…)

4. PROPOSAL FOR A EUROPEAN CURRICULUM FOR LOWER SECONDARY SCHOOL TEACHER TRAINING

Editing a unitary proposal for a European curriculum for secondary school mathematics teacher training is not an easy task, due to the deep differences between the participant countries' national training systems. In fact, it is enough to compare educational systems and teacher training systems of the project's partner countries (CZ, DK, FR, IT, SK) to understand how hard is to even think about a single European curriculum. One striking aspect concerns the different ages of involved trainee teachers: in some countries teacher training starts at the same moment as university courses, while in other countries it involves either graduates or people having a higher education degree (see Chapter 3).

More promising is the analysis, carried out in the five partner countries, of two extremely relevant and interesting aspects related to mathematics teacher training:

1. Lower secondary school curricula are basically similar as concerns mathematics. They generally share a long series of topics, except for some meaningful aspects. However, not the same can be said about the attainment of comparable objectives in terms of knowledge and competencies reached at the end of this school level.
2. Great similarities can be found in *educational methods for teacher training*. Going beyond differences in levels and formative educational paths, the different systems offer a number of proposals including both traditional lessons and activities involving trainees directly within an active approach to the different topics. There seems to be a shared attempt to guide students to perceive links between mathematics and reality as well as between mathematics and other disciplines.

Starting from these remarks, it seems possible to set up a shared proposal centred at the following two aspects[9]:

3. A set of *topics to be dealt with*, embracing all the topics characterising the different Lower Secondary School curricula in each of the participant countries (see Chapter 2). This list can be the basis for training activities to be proposed to trainee teachers. When trainees show to have attained competencies related to these topics in their previous education, the trainer is allowed to focus on epistemological, historical and educational reflections. Otherwise, these topics will be explicit object of teaching.
4. A meaningful *method for outlining proposals*, allowing pre-service teachers not only to acquire (or reflect upon) the different topics, but also helping them understand the potential difficulties their pupils will meet. The aim is to outline possible interventions or teaching strategies that may help pupils overcome these difficulties and increase their understanding.
5. This method, constructed on the basis of shared meaningful practices, was elaborated, experimented and documented by the Project's Team.

[9] The word *curriculum*'s etymology includes two aspects: the proposal's content and the instrument (*currus*) that enables one to both propose and make the content accessible.

The Project's Team chose some of the topics we referred to earlier, and identified some aspects of practice they considered to be positive and effective: then they carried out teaching experiments according to modalities we will illustrate later. Good teaching practice described in the next chapters sheds a light on methodology and, at the same time, offers the opportunity to evaluate potentialities and possible limitations.

TOPICS TO BE DEALT WITH

We mean to identify a shared set of *topics to be dealt with* within a Lower Secondary School mathematics teacher training course. As we mentioned earlier, the best option was to refer to the different mathematics curricula for this school level. Since teachers are obviously supposed to know the topics they will later teach, trainers will avail of a solid basis for the educational activities they will propose to trainee teachers.

As to trainee teachers' competencies related to these topics, they need to be complete enough for trainees to master them, even when the correspondent school curriculum requires a rather superficial knowledge by pupils[10]. As before, when trainees show they have attained competencies related to these topics in their past education, the trainer is allowed to focus on epistemological, historical and educational reflections. Otherwise, these topics will be explicitly taught.

It is anyway desirable to make trainees aware that different levels of competencies are required (to them and to pupils), through a meta-cognitive examinations of what they have learned, in view of their forthcoming teaching activity.

The Project's Team did not meet major difficulties in choosing topics and making a list, because they referred to the topics already listed in the different national curricula: topics are actually the same ones, in the great majority of cases. In fact, when clear differences came out, the team decided to follow a criterion of "majority" and meaningfulness in the choice, taking into account opinions made explicit by components of the Project's Team. This is also shown by the choice of activities for teaching experiments.

Major differences in curricula are those related to suggestions about ways of dealing with the different topics (respecting the different national approaches) as well as to abilities that should be attained with relation to the different topics. For this reason, we thought that a deep comparison was not appropriate and we only listed the topics, referring to the single national curricula for further details.

In the table below, for each topic there is a note highlighting the main similarities and differences. In cases where topics were dealt with at very different levels, we tried to

[10] For instance, it is fundamental for trainee teachers to be able to deal with and solve equations and inequalities of first and second degree, whatever the curriculum's request about this topic. Therefore trainee teachers will practice with symbolic calculations and will be able to better "master" problems, due to their abilities in approaching them algebraically. Also, it seems advisable that the main plane geometry theorems be dealt with, (for instance Thales', Euclid's and Pythagoras' theorems), although the curriculum might include only some or any at all: this will enable us to work on the concept of theorem and its relevance.

identify those aspects that could provide a common ground for a discussion involving all the different partner countries.

Required Knowledge	Notes
Arithmetic • Integer numbers and operations; divisibility. Least common multiple and greatest common divisor. • Relative numbers and operations. • Fractions and operations. Decimal representation of numbers. • Rational numbers and Operations. Powers and Roots. Real numbers • Percent, ratios, proportions. • Scientific notation in powers of 10.	This is probably the topic with less difference: curricula agree on both required knowledge and abilities. Another common aspect is the suggestion about working with calculators, which become an object of suitable educational remarks.
Algebra • Using letters in formulas. Expressions with variables. • Linear equations and inequalities. • Examples of algebraic calculus.	The level of both knowledge and competencies required for this topic is extremely variable in the different countries, also depending on pupils' age. Some curricula introduce quadratic equations and inequalities and systems of equations. However the topic is generally dealt with through links to meaningful examples of applications.
Geometry • Point, straight line, plane. Half-line, line segment, half-plane, angle. • Circle, circumference of a circle. Polygons. Triangle. Quadrilateral. • Congruence and similarity of geometric figures. Isometry. Point symmetry. Line reflection. Translation. • Basic geometric constructions: angles, triangles, quadrilaterals, regular polygons. • Main 3D figures: polyhedron; cube; cuboid; prism; pyramid; circular cone and cylinder. • Coordinate systems: the Cartesian plane and reference systems.	This topic is not in the Slovakian curriculum, which nevertheless considers most of the abilities listed here as already attained at previous levels. Another variable is the required level of "theoretical" competence (knowledge of definitions, classification of theorems, Thales' or Pythagoras' theorems…), although it is generally agreed that it should be, at least partially, promoted explicitly. Some countries' curriculum promotes the use of suitable software together with classical instruments, for geometrical constructions. However, this is a widespread practice, even where it is not explicitly requested.
Functions • Functions and graphs. Linear and quadratic functions. • Direct and Inverse ratios; their representation.	

Required Knowledge	Notes
Sizes and Measurements • Measurements: meaning and computation. Units • Measures by formulas: surface of plane regular figures; side surface and volumes of some solids. • Angle sum of polygons. Length of circle. π • Scales.	Not all curricula explicitly list this topic, but they all require the acquisition of competencies related to measure.
Representation and organization of data • Data collection; representations and readings. • Frequencies. Bar charts, pie-charts. Averages.	This topic is not included in all curricula, but references to it can be spotted around in different sections. For this topic, it is generally recommended to refer to data related to meaningful real life examples and there is an explicit suggestion about using spreadsheets and calculators.
Problem solving Translating from natural into formal language; Using induction, generalization, deduction. Conjecturing, discussing and proving about observations in different contexts. Examples and counterexamples. Recognizing problems, data and goals. Formulating problems, describing procedures and giving solutions in an understandable manner, both in writing and oral. Critically evaluating the different strategies to solve a problem.	This topic is explicitly listed by the Italian and Danish curricula only, but it is implicitly used by many others and it was considered as particularly interesting by the Project's Team. The topic should be dealt with in an interdisciplinary way, making links to the study of other disciplines, both scientific and linguistic/humanistic.

Table 19. Topics: similarities and differences

METHOD FOR OUTLINING PROPOSALS

One fundamental issue for lower secondary school mathematics teacher training is that European countries may share a common teaching *method for outlining proposals*, providing a common basis for understanding, meeting and exchanging, beyond the differences characterising educational and training systems. This method must enable trainee teachers to acquire or re-enforce their knowledge about the considered topics, but also prepare them to face possible future didactical obstacles in the classroom context.

In order to reach these aims, trainee teachers should not be presented with the various topics merely following the "transmission of knowledge" model. In this case, not only would the acquisition of knowledge related to the topics be more difficult, but also the trainee teacher might become convinced that a similar approach works with students as well, with possible serious consequences. If the trainer calls for trainees' intellectual abilities, the outcome might be a focus on theoretical aspects of the proposal by trainees themselves and a consequent oversight of meanings linked to reality.

In actual fact, in all standards it is shown how mathematics gives tools to act, choose and decide in daily life. They promote the development of logical thought, abilities of abstraction and both bi-dimensional and tri-dimensional visualisation, using formulas, models, graphics and diagrams.

The matter is to give pupils a scientific education necessary for a consistent representation of the world and for understanding their daily environment; they have to realise that complexity can be expressed with basic laws.

The detachment between mathematics and reality in students' perception is always a source of difficulties: at the school level we are dealing with, pupils are going through a developmental phase that leads them to the acquisition of abstract and rational thinking. Mathematical concepts are understandable as much as they are rooted in real life, by means of meaningful examples of possible applications.

The same teaching proposals offered to trainee teachers should have as many as possible common points with the activities they will actually propose to their pupils in class, including all aspects related to interdisciplinary connections and links to real life, when compatible with the topic under consideration.

Therefore, we can talk about "learning by modelling". Trainers communicate their own conceptions of mathematics teaching by putting them into practice in the sessions they deliver. In turn, trainees are then expected to implement in their own classrooms the sessions they have experienced as pupils. *Modelling strategies* differ from cultural strategies (where the trainer passes on a piece of information), from *demonstration strategies* (where the trainer transmits a teaching practice by implementing it effectively in his/her classroom) and from *transfer strategies* (where the trainer transmits referential knowledge about teaching and tries to harness the transfer phenomenon carried out by the trainees).

In this way the trainee teacher is not only led to reconsider mathematical concepts in depth, increasing his/her theoretical understanding and appreciating their relevance: he/she also has the opportunity to experience, at least partially, critical points, obstacles and solutions that are likely to come up in their future classroom-based activities.

A collective discussion should follow the phase involving the acquisition of knowledge of the topic. In this way trainees are allowed to share opinions, difficulties and discoveries and to outline meaningful teaching strategies and concrete ideas about the examined topic.

A subsequent phase must test what trainees discovered: the same activity is proposed in one or more pilot classes, reporting on pupils' reactions as well as on successful or unsuccessful learning, possibly making use of video recordings. The final step is a collective discussion where, under the trainer's guidance, trainees start from the teaching experiments carried out in pilot classes to compare hypotheses, reflections and initial teaching choices with what came out from piloting: this offers trainee teachers the chance to systematise the discoveries made throughout their work.

Part II

Best Practices

LUCKY NUMBERS

by Marie Hofmannová* and Jarmila Novotná*

INTRODUCTION

The following unit is a part of LOSSTT-IN-MATH project piloted in the CLIL course (Content and Language Integrated Learning, i.e. teaching a non-language subject through the medium of foreign language) at the Faculty of Education of Charles University in Prague (Novotná, Hofmannová, 2000). This two-semester pre-service teacher training course is aimed at students as of the third year of their studies. It is a seminar, 90 minutes per week, with many activities run in the form of workshop.

The course is led by two trainers, one specialised in mathematics education, the other in methodology of teaching English. During the LOSSTT-IN-MATH piloting experiments, the CLIL course was attended by fifteen trainees.

The course was originally designed for teacher training of prospective teachers of mathematics and English language. It is conducted in English. Regardless of this fact, also students – prospective teachers of other non-language subjects and foreign languages participate. This feature enriches the course in the multilingual perspective. The course combines educational theory and teaching practice, bringing students gradually from lesson observation, mastering subject specific vocabulary and CLIL specific knowledge and skills. This is followed by microteaching of peers based on a variety of materials (e.g. textbooks, student-made worksheets) and concluded by a teaching module in real school conditions.

Mathematical content covers mathematics for lower and upper secondary levels and reflects both mathematics taught in public system of education in the Czech Republic as well as some aspects of the bilingual experiment carried out in selected upper secondary schools. From the language perspective, the aim of implementing CLIL is to provide pupils with more exposure. What CLIL offers to learners of any age, is a natural situation for language development which builds on other forms of learning.

For the purposes of piloting LOSSTT-IN-MATH proposals, we selected such units that seemed to be compatible with our course content. The activity Lucky Numbers was included in the set of investigative activities proposed by Western Australian Mathematical Association and later modified to combine mathematics and foreign language teaching.

* Faculty of Education, Charles University in Prague, Czech Republic.

Main piloting

by Marie Hofmannová and Jarmila Novotná

ORIGINAL TEXT

"Choose a number. Square each of its digits and add the squares to get a second number. Square the digits of the second number and add the squares to get the third number. Continue this way to get a sequence. If the sequence reaches 1, the original number is called lucky. If not, it is called unlucky."

1. Mathematical topics for development

Arithmetic and application of algorithms

2. Aims

***For trainers*:**

- Guiding the trainees from theory to practice
- Making the trainees develop a lesson based on a problem taken from a mathematics textbook
- Providing instructions and feedback

***For trainees*:**

- Investigating solving/learning strategies
- Developing a lesson plan
- Preparing own teaching material
- Peer teaching
- Classroom teaching

***For secondary school pupils*:**

- Experiencing the teaching of mathematical content through the medium of English language
- Making problem solving more effective by discovering regularities
- Practising skills of addition and multiplication of natural numbers

3. Description of the activity

The training activities were planned in five stages, i.e. five weeks. Stages 1, 2, 3 and 5 were carried out during the CLIL course (45-minute sessions), stage 4 in the secondary school (a 45-minute lesson).

Stage 1 *The trainees*

- solve the problem and compare different solving procedures,
- discuss the necessary knowledge and skills for each solution (from the learners' perspective in both mathematics and English as a foreign language).

- Homework for stage 2: the trainees prepare the first draft of a lesson plan (for team/peer teaching).

<u>Stage 2</u> The trainees

- teach one phase of their lesson plan (team/peer teaching),
- analyse the teaching attempts,
- suggest changes and select the best ideas for the final draft of the lesson plan.
- Homework for stage 3: Group work – group 1 prepares the final draft of the lesson plan, groups 2 and 3 prepare the necessary teaching materials and aids.

<u>Stage 3</u> The trainers

- check and discuss the final draft of the lesson plan and the teaching materials and aids with the trainees,
- together with the trainees they select two student-teachers who will teach the lesson in a real classroom.

<u>Stage 4</u> At the secondary school

- the two student-teachers teach the 45 minute lesson. The remaining trainees and the trainers observe, take notes and video-record the lesson.

After the class:

- the student-teachers get immediate feedback from the learners (about 5 minutes),
- together with the other trainees and the trainers they discuss the course of the lesson (about 10 minutes).

<u>Stage 5</u> The trainees and the trainers

- watch the video-recording,
- reflect on the teaching experiment.

The trainers

- evaluate and assess the student teachers using the material for student-teacher evaluation and assessment during the teaching practice.

4. Assignments

a) Assignments for teacher trainees

- What prior knowledge is required for solving the task?
- Consider different starting numbers. What different stages do the sequences eventually reach? How many different stages are there?
- Look for ways of using one sequence to complete the others.
- Try drawing a diagram to show how numbers are related.
- Can you predict in advance any number that will be lucky/unlucky?

- What sorts of numbers produce sequences that are identical except for the first number?
- Try some three and four digit numbers.
- What proportion of the numbers for 1 to 50 is lucky/unlucky?
- Are lucky numbers more often odd than even?
- Investigate adding the cubes of the digits of numbers.
- Consider the previous mathematical tasks from the teacher's point of view.
- Discuss the first and third part of the proposal: What is the optimum student grouping?
- Would you extend teacher talk? How?
- What is the proportion between student and teacher work?
- What is the optimum timing for this activity? State variables.
- Consider task management aspects from the learners' perspective, e.g. systematicity, the proportion of oral/written work, division of roles.
- Mathematics taught through a foreign language: Write a translation of the assignment.

b) *Assignments for the pupils* (*presentation of the context*)

- Make a list of numbers considered by your family and friends as lucky numbers stating a variety of reasons.
- Results of our investigations show that different numbers are considered lucky for different people. Somebody's lucky number might be unlucky for somebody else.
- This, however, should not happen in mathematics. Let us define the lucky number as follows: "Choose a number. Square each of its digits and add the squares to get a second number. Square the digits of the second number and add the squares to get the third number. Continue this way to get a sequence. If the sequence reaches 1, the original number is called lucky. If not, it is called unlucky".
- Find all lucky numbers from 1 to 99.

5. Piloting

a) *In the training course*

A priori analysis of the text

- Discussing possible mathematical solutions.
- Anticipating methodology problems.

Preparing the lesson [*this stage was video recorded by one of the trainers*]

- The trainers and trainees discuss in Czech how to best prepare the microteaching of peers. They assign roles and prepare the first draft of lesson plan.

- Peer-team teaching in English: One stage of the proposed lesson is taught by two student teachers, the remaining trainees play the roles of pupils. One trainer takes notes on the blackboard for further discussion.

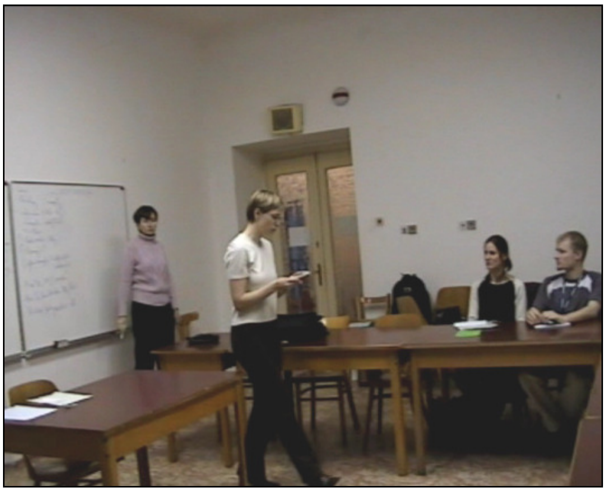


Photo 1. Peer teaching

- Trainers and trainees analyse the teaching attempts in English based on the notes on the blackboard. Comments and suggestions for the real classroom are proposed. The aim of the lesson is set for both mathematics and English.
- Mathematics – solving strategies.
- English as a foreign language – mathematical talk.
- Trainees divide into groups and decide who will prepare the final draft of the lesson plan and who will work on preparing the teaching materials (e.g. pictures, glossary of words) and they discuss the necessary aids.

b) *In the classroom*

The town of Kladno, lower secondary school, optional lesson, 8 pupils, 15 years of age, the classroom teacher, one of the teacher trainers, 45 minutes.

Teaching the lesson [*this stage was video recorded by one of the trainers*]

- The staffroom: Checking the lesson plan, materials, and aids.

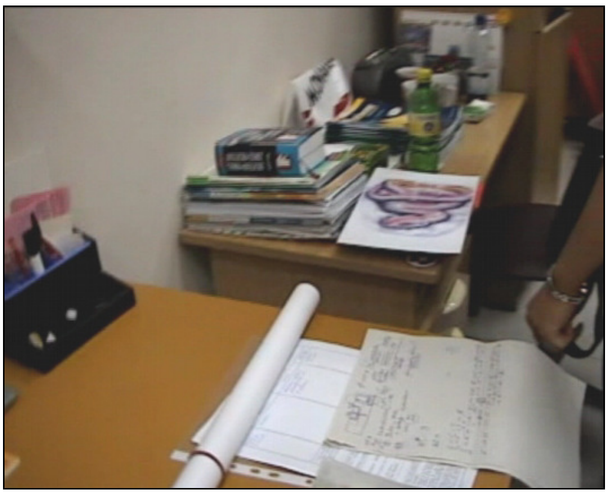


Photo 2. In the staffroom

- The classroom:
  - Introduction: The teacher motivates the pupils in English: good luck vs. bad luck.
  - The teacher presents visuals – six pictures. Pupils are asked to describe, the teacher elicits responses from pupils.

Photo 3. Using visuals

  - More information from pupils, unrelated to pictures, the teacher writes on the blackboard: lucky/unlucky numbers.
  - The teacher presents a problem: Is her date of birth a lucky or an unlucky number?
  - Listening comprehension: The teacher tells a story about a kingdom – lucky numbers.

Photo 4. Telling a story

  - The teacher introduces simple mathematical language in English.
  - Introduction of the procedure using 2 as the starting number (see Photo 5).

Photo 5. Switching into mathematics
(Example: $2 \rightarrow 4 \rightarrow 16 \rightarrow 37 \rightarrow 58 \rightarrow 89 \rightarrow 145 \rightarrow 42 \rightarrow 20 \rightarrow 4$)

- Controlled practice: The pupils and the teacher write on the blackboard. (A similar problem: February, i.e. 2^{nd} month, is it lucky or unlucky?)
- Free practice: The pupils carry out an individual pen and paper activity. The problem: “You proposed certain numbers as lucky for your families and friends. Will they remain lucky if you apply our procedure?” Feedback: Two pupils write on the blackboard.
- Further practice: individual work. The teacher poses the following problem: Is your date of birth a lucky or unlucky number?
- Revision of vocabulary in English: months. Further investigation linking to the mathematical topic: Is the pupils’ month of birth a lucky or unlucky number? The teacher elicits feedback: Individual pupils come to the front, write on the blackboard, and report back to the class.
- The teacher summarizes the class with the pupils using the table of numbers on the blackboard.

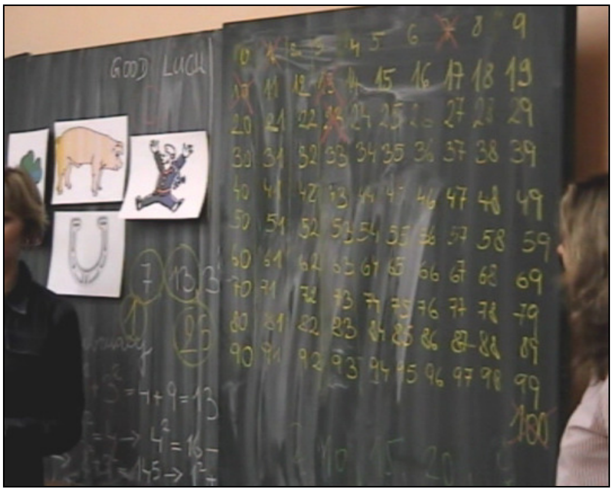


Photo 6. Table for the summary of results

- The teacher concludes the lesson.

c) In the training course

A posteriori analysis – reflecting on the lesson [*this stage was also video recorded by the trainer*].

The discussion was conducted in English and was fairly free. The items discussed were:

- lesson analysis
- comments
- critical remarks
- suggestions for alternatives.

During the discussion, the class spontaneously switched into Czech because both parties found it easier to express their feelings about the lesson in the mother tongue.

It was concluded that the experiment was the real success. Following to that, as a final point of discussion, one of the trainees decided to use the same materials and the lesson plan in order to carry out the lesson in a different secondary school through a different foreign language – Spanish. Her teaching attempt was also video recorded, this time by one of the trainees.

6. Concluding remarks

Comparing and contrasting the two video recordings enabled to make the trainees aware of the following facts:

- The personality of the teacher plays an important role because the lesson based on the same lesson plan with the same teaching materials might develop in a different way due to the different teaching style.
- Team teaching is advantageous for both the teachers and the pupils.
- Different foreign language of instruction did not constitute any obstacle to learning.

SUGGESTED READING

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Second piloting

by Jaroslava Brincková*, Iveta Dzúriková* and Pavel Klenovčan*

1. **Mathematical topics for development**

   Arithmetic and application of algorithms for age 13–14

2. **Description of the activity**

   Choose a number. Square each of its digits and add the squares to get a second number. Square the digits of the second number and add the squares to get the third number. Continue this way to get a sequence. If the sequence reaches 1, the original number is called lucky. If not, it is called unlucky. Distinguish notions to be happy and to have happiness. Investigate numbers in discussion club with the help of the Internet.

3. **Aims**

For trainers

- Guiding the trainees from theory to practice
- Providing instruction and feedback

For teacher trainees

- Mathematics: Problem solving, investigative mathematical procedures, generalisation.
- Methodology: Use methods of investigation in mathematics. Developing a lesson plan, peer teaching, classroom teaching.

For pupils

- Investigate numbers arranged by certain rules. Making problem solving more effective by discovering regularities.
- Practising skills of addition and multiplication of natural numbers.
- Investigate numbers in discussion club with the help of the Internet: http://www.pdf.umb.sk/moodle course/view.php?d=132

4. **Assignments**

See point 5 in the first piloting

In the assignments for teacher trainees the following task was added:

- Explain order of steps in solving problems in the scheme of Figure 1 below.
- Distinguish notions to be happy and to have happiness. Investigate numbers in discussion club with the help of the Internet: http://www.pdf.umb.sk/moodle

* Pedagogical Faculty, Matej Bel University, Banská Bystrica, Slovak Republic.

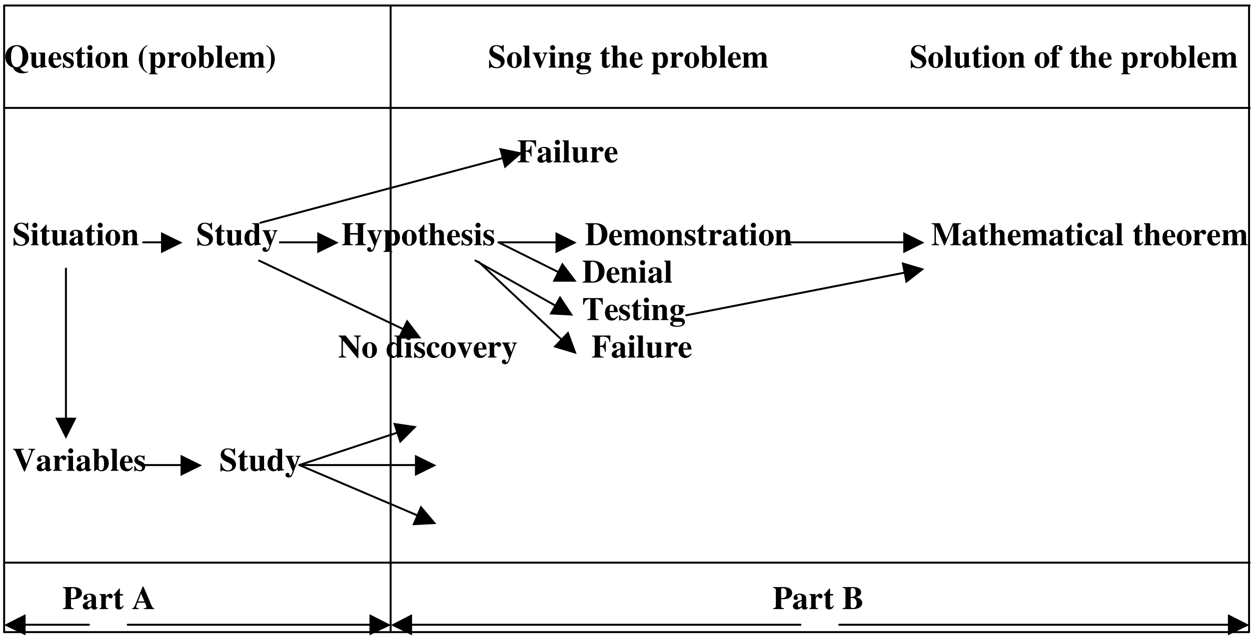

Figure 1. Problem solving procedure

5. Piloting

a) *In the training course*

Matej Bel University in Banská Bystrica, Pedagogical Faculty, Didactics of Mathematics course, Mathematics taught through English as a foreign language.

19 teacher trainees, 21- 22 years of age, 3 trainers, team teaching.

Time table: 45-minute training session, 5 successive weeks.

A priori analysis of the text

- Discussion about possible mathematical solutions.
- Preparing the lesson [*video recorded*]
- Discussion in Slovak [L1]: preparing the microteaching of peers. Assigning roles, preparing the first draft of lesson plan:
- Peer-team teaching in English [L2]: The teacher of English on lower level of 8-year grammar school has realised the lesson on topic My lucky number. The students have analysed video record with stress on mathematical communication in English.
- students teach, the rest role play the pupils, one of the teacher trainers takes notes on the blackboard
- Analysis of the mock lesson [L2] with the help of notes on the blackboard. Comments and suggestions for the real classroom, etc. Setting the dual aim of the lesson: Mathematics – solving strategies, English as a foreign language – mathematical talk.
- Preparing the lesson plan – stages [L2].

b) *In the classroom*

The town of Banská Bystrica, Evangelical 8-year grammar school, on lower level, optional lesson, 28 pupils, and 13–14 years of age, the classroom teacher, one of the teacher trainers, 45 minutes.

Teaching the lesson [*video recorded*]

- The staffroom: Checking lesson plan, materials, aids.
- The classroom:
  - Introduction – Teacher motivates the pupils [L2]: good luck vs. bad luck.
  - Visuals – six pictures: description, teacher elicits responses from pupils.
  - More info from pupils, unrelated to pictures, teacher on blackboard: Lucky/unlucky numbers.
  - Teacher presents a problem: Is her date of birth a lucky or an unlucky number?
  - Listening comprehension: Teacher tells a story about a kingdom – lucky numbers.
  - Teacher introduces simple mathematical language [L2+L3].
  - Introduction of the procedure.
  - Controlled practice: pupils and teacher on blackboard. A similar problem: February, i.e. 2nd month, lucky or unlucky?
  - Free practice: students – individual pen and paper activity. Problem: Are the numbers elicited in step 3 lucky or unlucky? Feedback: 2 students on blackboard.
  - Further practice: individual work. Problem: Students' date of birth – a lucky or unlucky number?
  - Revision of vocabulary [L2]: months. Further investigation: students' month of birth – a lucky or unlucky number? Feedback: individual students at the blackboard, report back to the class.
  - Summarizing: rule discovery using the table of numbers on the blackboard.
  - To be lucky – does it mean "to be happy"?
  - Eligible problem: Investigate numbers in discussion club with the help of the Internet: http://www.pdf.umb.sk/moodle/course/view.php?d=132
  - Concluding the lesson.

c) *In the training course*

A posteriori analysis – reflecting on the lesson [*video recorded*]

Free discussion [L2]: lesson analysis, comments, critical remarks, suggestions for alternatives.

Free discussion [L1]: teacher trainees express their feelings about the lesson.

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Kopka, J. (2004). Výzkumný přístup v matematice. Ústí n. Labem: Acta Universitas Purkyniane.

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Third piloting (at the University of Parma, Italy)

by Lucia Doretti*

The activity was coordinated by Daniela Medici and Maria Gabriella Rinaldi from the University di Parma, within SSIS (Specialisation School for Secondary Teaching) during classes devoted to the “Theory of didactical situations” and developed within three meetings:

First meeting (two hours): presentation of the problem situation to trainee teachers and successive work in groups for the analysis of the problem and identification of solving strategies. Discussion of proposals. Request, for the next meeting, of editing individually an a priori analysis in view of the classroom experimentation.

Second meeting (one hour): discussion on proposals emerged from a priori analyses and identification of classes for experimentation.

Third meeting (one hour): presentation of experimentations and discussion.

Notes and comments

For pupils from the classes

It was an experience that led them to look at natural numbers with different eyes, that is as “objects” with properties to be found and regularities to be studied. The activity, proposed as a game and a challenge, allowed pupils, working in small groups, to be directly engaged and raised interesting remarks to “fasten” the search for lucky numbers. Becoming aware of the problem situation stimulated pupils to search for lucky numbers with three or more figures and made them curious to get to a rule to find lucky numbers, they thought should necessarily exist. It was quite disappointing to come to know that such a rule is not known in mathematics, whereas they expected teachers to reveal it at some point. Pupils were led to reflect on the activity, the value of which lies exactly in the fact that they got autonomously, through discussion and exchanges, to new discoveries and “meaningful” reflections on the properties of numbers, although they did not have a traced path available. Almost all pupils, including those who were generally less motivated, participated in the activity, even

* Department of Mathematics and Computer Sciences, University of Siena, Italy.

though some were still puzzled about the usefulness of the work, since they "had not learned any new calculation rule".

For trainee teachers

It was the chance to reflect on the fact that investigating the multiple features of natural numbers can not only be curiosity, fun and play but also constitute a first step towards mathematics as search and discovery.

Trainee teachers, facing the problem of searching for lucky numbers, lived the same experience as their pupils: in front of a new problem, they need to explore its mathematical aspects and seek suitable strategies. They realised how useful this phase is to design and manage the activity in the classroom.

Planning a classroom intervention first individually and then through collective discussion made all the trainees feel involved and helped those who materially managed the implementation of the teaching experiment, as confirmed by the a posteriori analysis. In telling the story of what happened in the classroom, the experimenters pointed out the importance of and difficulties in managing the first phase in which the pupils had to get to understand and correctly use the "definition" of a lucky number. Overall they viewed the experience as an interesting opportunity for "experiencing how an a-didactical situation captures pupils' attention and stimulates their curiosity thus creating in the classroom conditions necessary to develop an interesting mathematical activity".

For trainers

The activity fell into the "Theory of didactical situations" domain, as an example of a-didactical situation, and was considered interesting by trainee teachers. Everybody seemed to be engaged, due to collaborative work and collective discussions, since nobody, including the trainers, had a "recipe" or the "solution" to get to. Many remarks were made and work was carried out in a very "constructive" atmosphere.

Conclusion

by Marie Hofmannová and Jarmila Novotná

One of the key issues in teacher education is a way how to set and maintain optimum balance between theory and practice, i.e. between the prospective teachers' knowledge on the one hand and their practical professional skills on the other. Over the years, there has been a lot of discussion on this problem with a number of different educational models compared and contrasted, their results analyzed and evaluated. The ideal solution, however, has not yet been agreed upon or found.

All teacher trainees in mathematics who enroll at faculties of education or other teacher training institutions have attended mathematics courses at primary and secondary schools. They have gained not only more or less extensive and deep awareness of notions and mathematical competences, but also a very personal experience of being taught. Teacher's prior experience can have significant influence on his/her ability of empathy as regards the cognitive processes of the pupils who often face the need to deal with new, often surprising notions, their properties and relations.

New educational materials put the stress on experimentation, data collecting, observation, rules discovery, generalizing, and hypotheses testing. Taking into account diversity of learning styles, such teaching strategies also promote individual approach towards the process of education.

The proposed and piloted activity called *My Lucky Number* is aimed at future teachers of mathematics attempting to link their mathematical knowledge and skills with the professional ability to teach the subject. As is shown in the three piloting events, for the teacher trainees it offers both the opportunity to simulate situations that can happen in the classroom, and a fair chance to reflect on their attitudes (conscious or non-conscious) towards mathematics and the way it is to be taught.

The original aim of the activity was to develop the pupils' ability to investigate in mathematics and to make use of the found properties of mathematical objects for further discoveries of object properties. Three piloting events of the activity called My Lucky Number have shown at least some of the paths how to develop a simple mathematical situation for a variety of teaching aims and objectives. Let us now look at the main differences of the three events. The common features have become apparent through the description of individual events, and we have it that it is not necessary to go into any more detail. To point out the differences will certainly be more interesting for the readers.

In the first piloting that was carried out by the authors of the proposal, particular attention was paid to the possibility to implement the activity in the course preparing future teachers for Content and Language Integrated Learning (CLIL), i.e. teaching mathematics through English as a foreign language. That is the reason why strong emphasis is put on the development of pupils' knowledge and skills both in general English and mathematical terminology. The preparatory stage in the teacher training course and also the real classroom teaching reflect this need. Individual preparatory steps aim at balancing the two main aims and creating meaningful links. The results of the secondary school piloting and follow-up discussions with the teacher trainees who participated in all the stages, testify to the fact that the activity made it possible for them to relate knowledge of both English and Mathematics (that they study at the university) with professional teaching skills necessary for the teaching of mathematics, English, but mainly for mathematics taught through English.

The second piloting was inspired by the originally designed scheme that was implemented in the first piloting, and drew on it significantly enriching it by two new ideas: reflecting on possible intersections in the solving procedure while dealing with (not only) mathematical problems, and using the Internet for mathematical discoveries. The proposed activity turned out to be a good example both in teacher training and school mathematics teaching. Especially the Internet discoveries seem to be a new and interesting experience for a number of prospective teachers who look for new paths in their professional development. Furthermore, it might become a strong aid in motivating pupils to an active approach towards learning.

The teaching/learning process can be characterized as a sequence of situations (natural or didactical) that result in modifications in the pupils' behavior that are typical of getting new knowledge[1]. Peculiar to this process are the so called a-didactical situations where the teacher passes some of the responsibilities for the learning process onto his/her pupils. On the teacher's part, it actually means delegating power, for the pupils it means gaining control. The pupils themselves, without the teacher's direct intervention, are investigating and discovering, they are creating a model and checking its correctness and usefulness, or they are creating a different model that they consider more useful etc. Their activity is controlled only by the learning environment and their knowledge, not by the didactical activity of the teacher. Each pupil becomes responsible for getting the required results. The teacher's task is to both facilitate such situations and institutionalize the information obtained by the pupils. The knowledge is further utilized and developed with the teacher's help. The third piloting showed how to use the activity called *My Lucky Number* to prepare future teachers for designing a-didactical situations and their development. Even here the results were excellent.

Summarizing experiences from the above described piloting events, we can state that discovery strategies with all their subcomponents play a very important part in teacher education. The same is true when using discovery strategies with school learners. Our experiences confirm, that those students who get acquainted with such strategies during their studies, will be more open and prone to using them later in their own teaching. Moreover, they will not be afraid of difficulties they can encounter while working with pupils, they will be more open accept individual learning styles etc.

The proposed activity was by no means intended as the only one that can offer help in the above sense of the word. It is just one example of a certain type of useful procedures, and at the same time a challenging path to follow in teacher training. It is necessary to always bear in mind, that the main aim of such activities, tasks or assignments such as *My Lucky Number,* is not to elicit satisfactory answers to the teacher's questions, but to assist the pupils to compare their prior knowledge and ideas with their own new discoveries, and also with the ideas and results of the others. The teachers' task is not easy. They need to uncover principles of gaining knowledge in order to prepare such didactical situations that enable the pupils accept responsibilities for their own learning.

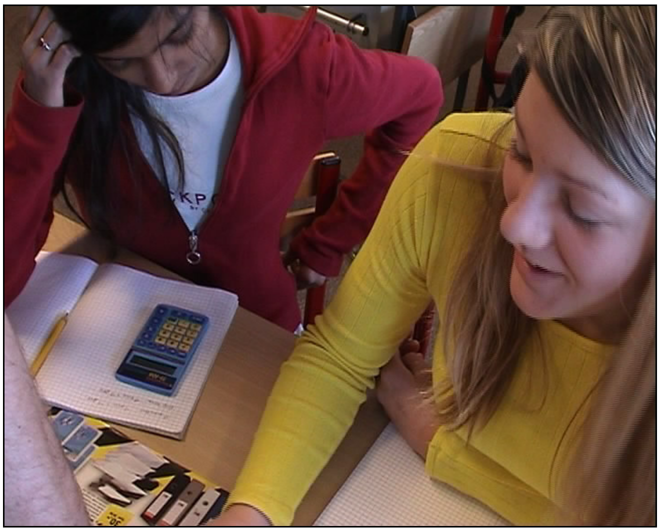
[1] Brousseau, G. (1997). Theory of Didactical Situations in Mathematics. [Edited and translated by N. Balacheff, M. Cooper, R. Sutherland, V. Warfield.] Dordrecht,/Boston/London: Kluwer Academic Publishers.

CELL PHONES

by Annette Jäpelt*

INTRODUCTION

The didactic proposal Cell phones presented here is a contribution to the LOSSTT-IN-MATH project. It consists of a comparison of different tariff plans available for mobile phone calls. The subject has been chosen, because it is a major part of pupils´ every day life and because it is a rather complicated subject with many variables. In treating this, the competences of problem treating and modeling can be developed. In the proposals the tariff plans will be looked at through mathematics notions. The proposal was piloted by the following partners: Skårup Seminarium, University of Pisa and IUFM of Paris. For the first two partners the following scheme was contained in the proposal: Introduction for teacher trainees. In groups teacher trainees are discussing how to make the best lesson plan for pupils. A lesson plan is made. Teacher trainees are piloting the proposal in school.



* Skårup Seminarium, Denmark.

Main piloting

by Annette Jäpelt

PROBLEM SOLVING WITH BROCHURES OR COMMERCIALS

The following unit is a part of LOSSTT-IN-MATH project piloted in Skårup Seminarium in the third year of teacher training in mathematics.

The teacher education takes four years and the teacher trainees are authorized to teach at primary and lower secondary schools.

Mathematics is a part of all four years, within the same team and normally with the same teacher trainer. Each week the teacher trainees are offered about five hour in this subject.

PROPOSAL

It is important that the pupils can manoeuvre in their daily life. There will be a lot of areas where mathematics is the only subject that can contribute to this. The pupils are bombed with commercials concerning mobile phones and they often use a lot of money on these. This subject really affects the pupils' daily life and is very essential for them. At this age (13–14 years) they often administer shopping and payment themselves. Ann Tariff plan for mobile phones contains a lot of variables, which makes it very difficult for most people to determine what will be the best buy for their own needs. That is the reason why the didactic proposal "mobile phones" was chosen for the project. The teacher training students are asked to prepare a plan about this problem to be carried out in a lesson with pupils in the seventh class. The pupils should be provided with the necessary data, which are usually contained in a commercial. Looking at linear functions would be the most obvious thing to do. But the class has not worked on the notion of function and then the most obvious thing is to work on the notion of variable, as we have too little time to introduce the pupils to the notion of linear function.

The plan of the piloting in Skårup is shown in the following table**:**

Main persons involved	Chronology	Description
Teacher trainer		The teacher trainees are sent an introduction via e-mail.
Teacher trainees	1hr	One teacher trainee has prepared an introduction, based on the information from the Internet. He informs the other trainees about prices concerning mobile phones.
Teacher trainees in groups	1hr Video recorded	The teacher trainees work in groups to plan the lesson with the pupils.

Teacher trainees	2hrs Video recorded	The whole team discusses how to make the lesson including the conditions for the lesson with the pupils
Pupils and six teacher trainees	1hr Video recorded	The lesson at school
Teacher trainer and teacher trainees	1hr	Evaluation

Table 1. Piloting plan

General information

Number of trainers: 1 (a teacher at Skårup Seminarium)

Number of trainees: 25

Number of classes involved in the piloting: 1 (7^{th} class)

Number and age of pupils: 20 pupils aged 13-14

Number of adults in each classroom during the lessons: 5 trainees (present for the first time in the classroom) the teacher and the teacher trainer.

Aims

For trainers:

- Providing information
- Helping the teacher trainees from theory to practice
- Evaluating

For trainees:

- To treat problem solving as a part of mathematics and under that to involve daily situations in mathematics education.
- To understand and use expressions, which include variables
- To discuss and develop a lesson plan
- To implement the lesson plan

For pupils:

- To treat problem solving as a part of mathematics
- To handle daily situations in their mathematics education
- To work on the notion of variable
- To manoeuvre in their everyday life
- To take a critical stand on the information they get in the advertisements

Instructions for the teacher trainees

As an introduction the following sheet was sent to the teacher trainees by e-mail:

Problem solving in connection with brochures and commercials

Subject: Mobile phones

Aims: As presented above

Description: The pupils in the seventh class are presented with a problem where the necessary data for the solution of the problem are contained in a commercial.

Instructions for the pupils: The pupils receive several commercials and are asked questions about them. They have to make use of mathematics to come up with an answer. Then they are asked to make their own, maybe more descriptive, commercial.

Assignments for the teacher training students:

- How can you organize the lesson for the pupils?
- What questions should be asked?
- What precisely do you want the pupils to do?
- How many commercials would you show the pupils or should they possibly find some themselves?
- Are the pupils better at solving real problems than solely technical ones?
- Which difficulties do you anticipate in connection with the lesson?
- What do you expect the pupils will get from this lesson?
- What sort of performance can be evaluated and how in this lesson?
- Which similar problems can be treated in this way?
- Further: You may bring some mobile phone commercials yourselves.

THE PILOTING

The course of the lesson with the teacher training students

The number of teacher training students is 25. They study in the third year at Skårup Seminarium to become teachers. One teacher trainee has prepared an introduction to mobile phone prices, on subscriptions, on voice charges and on text messages. This introduction lasts the whole lesson.

Preparing the lesson [*This part was video recorded*]

The teacher trainees worked in groups to plan the lesson, which also lasted an hour. Fragments from the presentations of the teacher trainees' group work are reported in the following.

<u>Group1</u>

As a loan or buying. Two different subscriptions. Which one will they choose?

<u>Group2</u>

Different advertisements and three different requirements. Each group is going to have one advertisement and three requirements. The group is to prepare an algorithm for the price and a graph. After this the groups are going to compare the prices. They objectify it to get the text transformed into something useful.

Which one is best for you?

It looks like a storyline.

If we had more time the problem could be more open.

<u>Group3</u>

The pupils are shown a lot of advertisements.

The pupils are going to work with their own requirements, and arguing for their choice of telephone.

The pupils should be placed in groups, where they are going to prepare graphs of their requirements for the different telephone companies. After this they should prepare their own advertisements.

In speaking: the groups of pupils are going to argue for their choice of telephone in plenum.

Photo 7. Teacher trainees present their groupwork

Finally the whole team discusses how to make the lesson including the conditions for the lesson with the pupils:

- The teacher training students do not know the pupils.
- There is only one lesson available and no time for reflection.
- The pupils have not learned about the notion of function.

- The teacher training students decide to choose both subscription and need for voice and text because of the limited time together with the pupils. They also select to control the lesson very tightly.

Fragments of the plenary discussion:

- *Everybody wants the pupils to make a graph.*
- *Because not everyone has a mobile, the pupils should have fictitious requirements.*
- *Which factors should be kept as variables?*
- *Is there time for the pupils to be critical?*
- *Final selection.*
- *Dialogue about prices in general.*
- *Fictitious requirements and a graph presenting this.*
- *How are they going to draw the graph?*
- *How much work are the pupils able to do in the given time?*
- *Each group gets one advertisement.*

The introduction they made is shown underneath (the students' introduction to the pupils).

Further the pupils must draw a bar chart on a piece of paper by hand to visualize the situation.

Finally the teacher training students need time for discussion.

Five students volunteer for teaching the pupils. One of them wants to make the introduction and there is then a student each for each group of children, as the students' plan that the pupils should work in five groups of four.

INSTRUCTION WITH THE PUPILS

As the teacher training students have relatively limited class time they choose to guide the pupils by having a relatively fixed introduction. The children are still shown the commercials, but do not need to select the information themselves.

The teacher training students' introduction to the children:

Mobile mathematics in the 7th class

At first, a common introduction in the class.

The class is divided into 5 groups:

- Group 1: Company one
- Group 2: Company two
- Group 3: Company three
- Group 4: Company four
- Group 5: Company five

Each group gets 3 fictive needs and one subscription. From this they should calculate which need best suits their subscription.

Need 1: 10 minutes voice + 600 text messages. (blue in Picture 1)

Need 2: 1 hour voice + 200 text messages. (red in Picture 1)

Need 3: 3 hours voice + 60 text messages. (green in Picture 1)

Each group produces a bar chart showing 3 different needs according to their subscription.

Finally we will possibly discuss the price differences in the class.

Example of what is expected as the pupils' outcome of the lesson

To provide understanding the following example showing calculations and column diagram is inserted.

The price at company two is:

- 0,75 DKR/min voice
- 0,20 DKR/text

From these data the pupils should calculate which subscription suits their needs best.

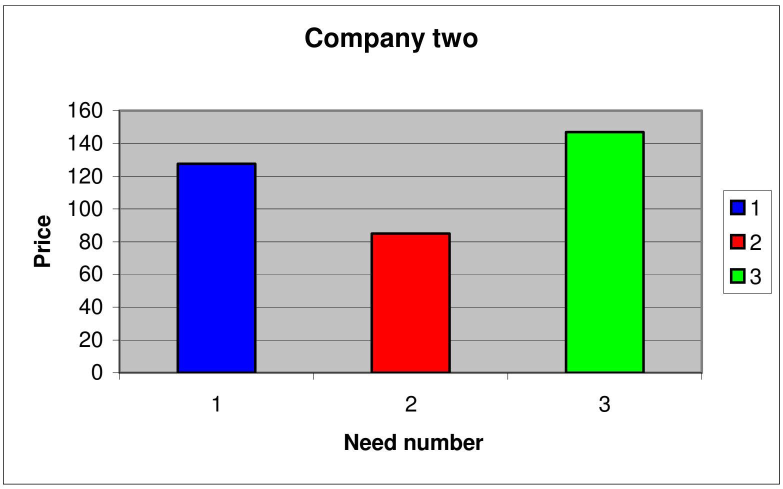
The prices for group two would then be:

Need 1: 10 minutes * 0,75 DKR/minute + 600 texts * 0,20 DKR/text = 127,50 DKR

Need 2: 60 minutes * 0,75 DKR/minute + 200 texts * 0,20 DKR/text = 85,00 DKR

Need 3: 180 minutes * 0,75 DKR/minute + 60 texts * 0,20 DKR/text = 147,00 DKR

The bar chart will look like the one in Picture 1.



Picture 1. Bar chart showing final costs

Progress of the lesson

There are twenty pupils aged about 14 years. The school is Øster Åby Free School situated a little north of Svendborg. The math teacher of the class is present but only observing. Present were six teacher training students, one made the introduction and one each helped each of the five groups. The pupils were concentrated on their work. The corporation between the pupils and the teacher training students was good and the teacher training students did not take over the initiative but let the pupils be active and put helping questions to assist the pupils to come a step forward if they stopped. The pupils calculated how much their assigned subscription would cost with the three given needs. Further the pupils quickly drew the planned diagrams. In this way they got time to look at some of the other subscriptions, which gave them a good basis for the following plenary discussion.

Finally the pupils discussed with the teacher training students which subscription out of five they would choose and further which subscription they had actually chosen themselves and the reason for that.

Photo 8. Pupils work in the classroom

Feedback to the teacher trainees

At the end of the lesson with the pupils the trainees were asked what they thought about the lesson and they said it had been good.

The mathematics teacher of the class said that the pupils had spoken positively about the lesson.

To have a basis for a dialogue I had handed out an evaluation scheme to the teacher training students. This scheme can be seen here including some of the remarks from the teacher training students.

Evaluation of the lesson about mobile phones

All students were asked to fill in all points for the lessons in which they had participated:

Task	What was good?	What was bad?	Remarks
Preparation for the children	*Initial preparation in small groups bringing more input to be presented in the “big” group.*	*Maybe somebody did not express their ideas and just agreed with the others?*	*Possibly make an agreement beforehand about how the*

	Good having somebody to discuss with. *Good group work creating many ideas.* *Fine progress in which everybody brought their ideas.*		*final selection will take place.* *Let every "small" group write a page with its proposal*
The pupils' learning	*Good to work in groups, but there is always somebody just "hanging on".*	*We should make a common agreement on what to do when the fast ones are finished.*	*Some finished very fast but others not.* *Difficult to differentiate and make sure that everybody understood.*
What did you learn from the lesson?	*Experience in planning and accomplishment.*		

Table 2. Evaluation scheme for teacher trainees

Further the teacher training students were asked how the evaluation of the pupils could look like. One of the answers looks as follows.

I think that the pupils must only evaluate the elaboration of the instructions and not if they have got a technical result, which I believe should be measured through similar problems.

The evaluation could be:

What is it like to work in groups on mathematics? Why?

How do you prefer to work? – Why?

What is it like having so many teachers around? – Why?

The evaluation discussion takes place in the class 10 – 15 minutes later within the same groups as the children worked in.

Description of another teaching practice situation

At the beginning of the study year 2004 we worked with linear functions at the teacher training institute. As one of the many specific examples we chose to work with payment in connection with mobile phones.

Later one of the teacher training students used this example in her third year of the teaching practice. I visited her in the class when they treated this subject and recorded the lessons on the video. The lessons took place in the sixth class in one of the schools in Svendborg.

All pupils learned something within the two lessons, but the differences were very big. Some pupils could draw the diagrams for two different subscriptions with help; others could not draw a diagram even for the simplest linear function, but were then able to take the first step towards the variable expression. Most of the pupils were in the middle group between these extremities. If the pupils should work with this before treating linear functions the situation must be very simplified:

It should only concern whole numbers and only one variable.

This topic should possibly be treated later but may contribute to a developed education differentiation as it contains a lot of possibilities in what you may include in the calculations.

COMMENTS

The teacher training students decided to guide the lesson fairly tightly because of the limited time available.

The pupils were good at carrying out the operations they were asked to and were also able to convert the results to be used in reality.

One of the factors I, as a teacher trainer, want to strengthen is to develop the pupils' ability to read a text and make a mathematical model out of the text. Unfortunately we did not have time enough to carry out this task. This would be the first step in using mathematics in daily life. A general question in this connection would be if pupils were able to transfer subjects learned in school mathematics to daily life. I often experience that teacher training students separate knowledge learned in one subject or even part subject from what they learn in other places and that the transfer value from one part of their life to another is even very small.

If the pupils often could use mathematics on relevant problems, the engagement and with that the ability to remember and use mathematics would increase. With this also the possibility that the pupils when they have finished school could use mathematics in their daily life would increase.

Even within the school system I have many times experienced that it looks like all knowledge is lost. I have seen that students in upper secondary school seldom were able to use their school mathematics and following I have experienced that our students at the teacher training institute were not able to use mathematics learned during their upper second education. Maybe they would have remembered the mathematics if there had been a higher engagement and less teaching by routine. Routine work is needed, but it should not be the only and dominant way.

A comment about the subject

Because of the great number of variables it would be more relevant to use spreadsheets.

In Denmark it may be the following variables:

> *Phone cost – Monthly charge – Minute charge – Call charge – Video charge – Text charge – Included charges*

The price is a function of many different variables, which makes it very difficult for most people to determine what will be the best buy for their own needs.

With a spreadsheet you can observe very quickly what happens when we change variables and get a visual picture of what happens by such a change. For the pupils the picture of a change will often drown in time demanding calculations.

A general comment

Both for me as a teacher at a teacher training college and for the students it has been very fruitful to be so directly involved in planning the lesson in a school class. Normally trainers are rather involved as advisers than participants in the teaching practice of their trainees: the experience for me with this project means that I will try to change this. It has been very relevant to follow the process from the trainees' first learning through the preparation and to the teaching in the school. The trainees' engagement and reflection have been very great. More students than normally are active when they can see the direct relevance.

I wish that the whole teacher training is based on regular interaction between theory and practice.

ACKNOWLEDGEMENTS

Thanks to Øster Åby Free School, seventh class and their mathematics teacher, Brian M. Østergård.

Also thanks to my mathematics class 22.4 at Skårup Seminarium for their kindness.

Second piloting

by Franco Favilli* and Carlo Romanelli**

Within the framework of the LOSSTT-IN-MATH project activities, the didactic proposal *Cell phones* was prepared and presented for discussion to student teachers. The proposal was later piloted by two pairs of trainees in two lower secondary school classrooms.

Among the mathematical notions necessary for the comparison, special attention was paid to proportionality, arithmetical progressions, functions, graphics, approximations and basic elements of statistics. During the classroom activities, pupils also made use of the Excel software.

* University Centre for Educational Research and Training – CAFRE, University of Pisa, Italy

** Istituto Comprensivo "E. Pea", Seravezza, (LU), Italy.

The proposal

The *Cell phones* proposal seems to correspond with a good way of dealing with mathematical notions, introducing them through a nice mixture of theoretical and practical activities. Expanding and deepening could easily lead the discussion well beyond the standard content of a lower secondary school mathematics curriculum. Its piloting requires therefore, first of all, the definition of specific didactic aims and the selection of only a few of the possible mathematical notions for introduction or further use (if they are already available to pupils).

The trainees work in twos or threes. The trainers give each group several tariff plans for mobile telephones calls and ask them to compare different tariffs. Each group is then asked to identify the most convenient tariff plan for the average use of the mobile phone. All the different choices are finally presented and justified in front of the other groups.

The same scheme should be used both by the trainers with the trainees and by the trainees with the pupils at school.

The plan of the piloting was designed and developed in the following scheme:

Steps							
Trainers (10h) Preparation of the didactic proposal	*Trainers and trainees* (4h) Introduction Group work Discussion	*Trainees* (2h)	*Trainees and pupils* (2h) Introduction Group work Discussion	*Trainees* (2h)	*Trainees and trainers* (4h) Reporting Discussion	*Trainees* (4h)	*Trainers* (5h) Final report

Objectives		**In-context methodology**						
			Trainees	*Pupils*		*Trainers*	*Trainees*	
Short term	Knowledge Competences	Thinking it over Lesson planning	Knowledge	Knowledge Competences	Thinking the lesson over Report	Socialization		Thinking it over – Remarks Finalizing the lesson plan Report to trainers
Long term	Methodology Socialization		Methodology	Socialization		Methodology	Methodology	

Table 3. Plan of the piloting

General information

Number of trainers: 2 (a university and a lower secondary school teaching staff)

Number of trainees: 42

Number of classes involved in the piloting: 2 (3^{rd} class of lower secondary school)

Number and age of pupils: 40 pupils aged 13

Number of adults in each classroom during the lessons: 2 trainees (present for the first time in those classrooms) and the teacher

Aims

The educational aims of the proposal can be roughly divided in general and content aims.

Among the *general aims* we can consider:

- Making the pupils better aware of the use of mobile phones and the need of critical choices.
- Setting a questionnaire.
- Promoting cooperation in problem-solving activities.
- Increasing the knowledge of terms from economics, finance and commerce.
- Involving the pupils' families in the educational activity.

Among the *mathematical aims* we can consider:

- Making the teaching and learning of mathematics more attractive and effective through investigation and evaluation of daily life needs.
- Identifying a problem to be solved by a statistical survey.
- Starting from a real problem, introducing or consolidating basic statistical concepts, such as population, data resources, data collection, statistical units, data organization, representative and non-representative data samples, frequency and percentage, statistical indexes, their representation by and interpretation through graphs.
- Starting from a real problem, introducing or consolidating basic mathematical concepts, such as function, graph of a function, step function, arithmetical progression, direct proportionality, approximations, equation of a straight line.
- Improving basic technology education of the class, and the disadvantaged pupils in particular.

Assignments

For the trainees

Planning a unit always requires the teacher to make methodology and didactical choices. As regards the *Cell phone* proposal, its preparation by the trainers, its piloting with the trainees first and with the pupils after, the discussions during the training sessions, made both trainers and trainees aware of the relevance, for the piloting itself, of the different possible answers to questions and issues such as the following ones:

- What kind of tariff plan have you got for your mobile phone?
- Have you ever considered comparing your plan with others, recently introduced?
- Have you ever tried to evaluate the actual cost of your average call per minute?
- What does *average call* mean for you? Compare your meaning with other pupils in the group.
- Are you aware of the amount of variables that make such a calculation possible?
- Do you think it is easy to calculate such a cost?

- What kind of knowledge do you need to calculate that cost?
- Make a list of mathematical and statistical notions that, in your opinion, are necessary.
- Which of them could be found in a lower secondary school mathematics programme?
- Try to identify at least two tariff plans that could allow thirteen-years old pupils to calculate the cost of an average call from their mobile phone with such tariff plans.
- Make a comparison of the graphs (x = min, y = €) of two tariff plans.
- Maybe you need to fix some variables…
- Try to prepare a mathematics lesson plan for a lower secondary school class.
- What about submitting a questionnaire to the pupils to introduce them to the topic, thus making them aware of its complexity? You would have a better knowledge of the class context.
- Maybe you could start modifying a tariff plan, thus obtaining something quite simple.
- Are you considering the use of software such as EXCEL to process the different information and use this to draw the graphs?
- What kind of approximations will you make use of?
- What kind of links could you make to other school subjects?
- What kind of application of this problem-solving proposal could be possible in an upper secondary mathematics classroom?

For the pupils

- How much money do you spend on the mobile phone in a month? Make a comparison with your classmates.
- Do you know which tariff plan you have? If not, call your mobile phone operator and ask for the details of your tariff plan!
- Let us compare the information you have obtained about your tariff plan with this one (see Figure 1 below).
- Which of them is more convenient for your average use of the mobile phone?
- How long should an average call last to make your fare plan more convenient?
- Have you ever tried to make any similar comparison?
- Can you imagine that it is possible to make a more accurate comparison of the tariff plans with the help of mathematical notions?
- Will you look for a different fare plan?
- Try to identify a tariff plan that could better fit the needs of your group members.
- Draw a new tariff plan that would be ideal (and realistic!) for the whole group.

- Are you satisfied with the experience? Why?
- Why, in your opinion, did your teacher propose this activity to you?
- Make a report of the activity.

The training session

The trainees were grouped in pairs (see Photo 9). The trainers gave them several different tariff plans (available on the Internet) for the mobile phone calls for comparison. The trainees were then asked to identify the most convenient tariff plan for their average use of the mobile phone.

Photo 9.Trainees working in pairs

At first, the trainees identified different variables to be considered for the mathematization of the problem and made a list on the blackboard. The complexity of the problem emerged quite easily. After an agreement on the variables to be considered, thus making a partial use of the information available from the tariff plans and partly modifying them, their comparison was sketched on the blackboard, through Cartesian graphs.

Taking into account the university careers of the trainees (non-mathematical scientific graduation), the need for better knowledge, especially of certain mathematical and statistical notions was clearly seen. Therefore, the trainers provided them, on the spot, with basic information needed to carry out the task that the trainees were about to do.

The trainees were then asked to think about a possible structure of the lesson plan on this topic in a lower secondary school. After a long and profound debate, first in groups and later in the whole class, two pairs of student teachers agreed to prepare and pilot a lesson plan and to report their experience back to the class.

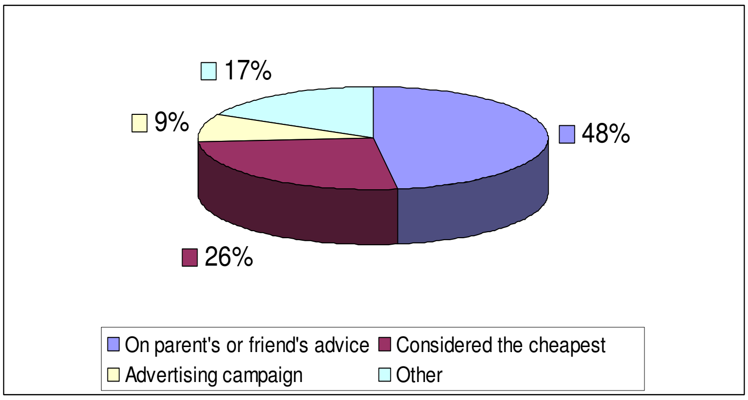
At the end of the session the trainers gave the trainees further tariff plans and examples of possible mathematical questions arising from their application.

The session in the classroom

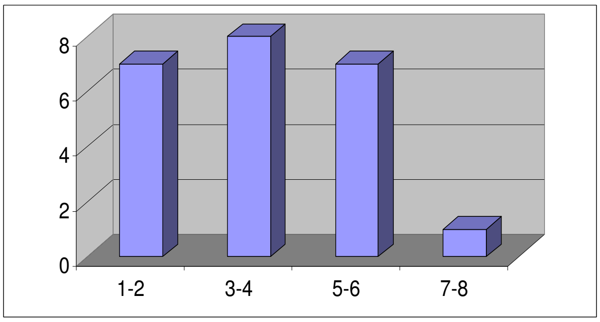
Before piloting the proposal at school, the two pairs of trainees met, thought over the activities in the training session and agreed to introduce the problem in a better, motivating way – by setting the pupils of two classrooms a questionnaire dealing with their use of mobile phones. Pupils were also asked about their parents' role in making choices about mobile phones and their behaviour. The analysis of the answers obtained in the classrooms proved how poor was the pupils' awareness of the actual cost of their calls and of the different variables influencing it. And what is more, how poor was their interest to look for a tariff plan that would be more suitable for their needs. The questionnaires analysis was supported by the use of Excel software.

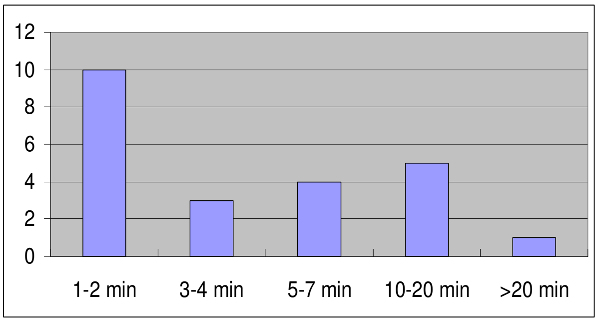
Later on, the pupils were given tariff plans that the trainees had adapted for the specific educational context so that they could be more easily interpreted and used for the problem-solving activity. The pupils worked in pairs.

Mathematical and statistical notions were used (see *Aims*) mainly in the construction of the graphs and the analysis of the different needs and habits shown by the pupils in their responses. Different types of graphs were used for the statistical analysis, as shown in the figures below:



Picture 2. Motivation for the tariff plan choice





Picture 3. Average calls per day/pupils

Picture 4. Average length of the calls/pupils

The role and importance of numerical approximations and different stages for their introduction when computing the costs necessary for the tariff plans comparison clearly emerged, as reported by a student teacher: some pupils have approximated the numbers at the end, after having made the computations always using non-approximated values, some others have soon approximated the numbers they had obtained and have made use of these approximated values for the following computations (see Photo 10). It was nice to show, during the debate in the classroom, how such differences could imply a different evaluation of the most convenient tariff plan.

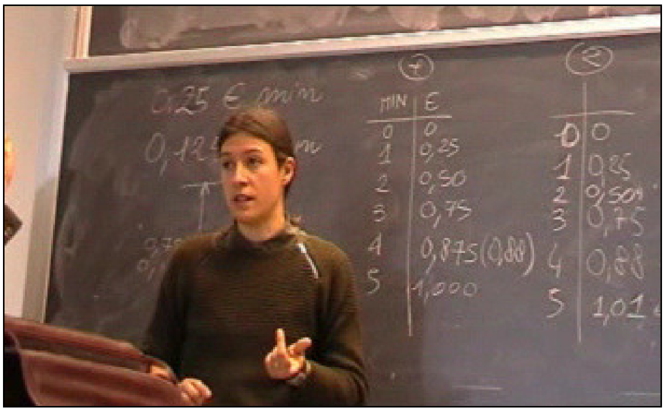
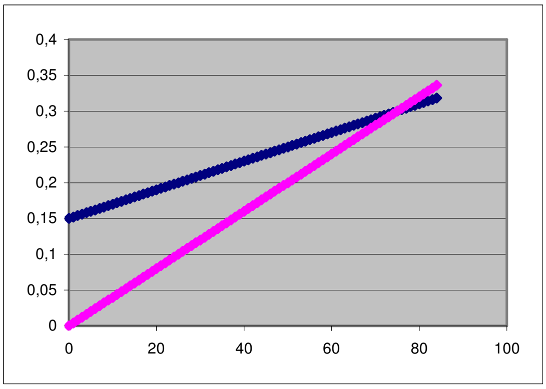


Photo 10. Approximations at different stages

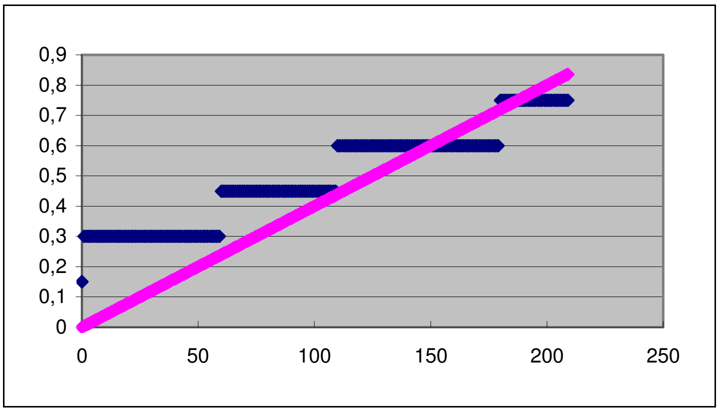
The piloting also showed that it is possible to introduce the notion of arithmetical progression even in lower secondary schools (see table and graph in Picture 5).

Minutes	Toll at the answer y = 0,15 + 0,002•x	No toll at the answer y = 0,004•x
0	0,150	0,000
1	0,152	0,004
2	0,154	0,008
3	0,156	0,012
4	0,158	0,016
5	0,160	0,020
6	0,162	0,024
7	0,164	0,028
8	0,166	0,032
9	0,168	0,036
10	0,170	0,040
11	0,172	0,044
12	0,174	0,048
13	0,176	0,052
14	0,178	0,056



Picture 5. Arithmetical progressions and their graphical representations

The notions of linear function and step function were illustrated through their graphical representations (see graph in Picture 6), thus making it possible for each pupil to better evaluate, in relation to the individual average use of the mobile phone, the advantage of choosing a tariff plan based, for example, on the proportionality between cost and time, instead of a tariff plan based on time units, or vice versa.



Picture 6. Comparing tariffs (linear function vs. step function)

It is important to mention that, as expected, some pupils found it extremely difficult to use Cartesian coordinates with different measure units for the x, y variables.

Feedback for the trainees

After the piloting, the four trainees met again to reflect on the experience and prepare the report for the other trainees. In view of this, they also watched the video recordings made during the classroom activities. To prevent pupils from feeling uncomfortable with another adult in the classroom, the lessons were video recorded by one of the pupils.

It was agreed that, besides the two trainers and all the trainees, two pupils would participate in the feedback session.

The four trainees presented the school piloting, made comments and remarks, and showed the most important video-clips taken in the classrooms. Most of the above described outcomes from the piloting were introduced for discussion (see Photo 11).

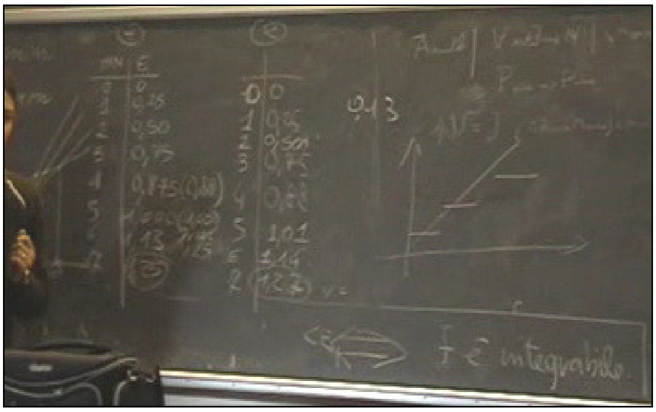


Photo 11. Tables and graphics from the trainees reporting

Unfortunately, not so surprisingly, while the two pupils were active in the discussion, the non-piloting trainees only occasionally entered the debate.

It is a matter of fact that the two pupils soon seemed to feel comfortable in the new (for them) educational context. Apart for some interesting comments and general appreciation of the proposal they had actively contributed to pilot, the pupils made it explicit that they were all surprised, in the class, by the large amount of mathematics involved in the supposed-to-be easy problem and, as a consequence, by its complexity. As was expected, the trainees acknowledged the possible use of the proposal in an upper secondary school. This would enable the introduction of further mathematical notions and the production of examples of problems in the field, for example, of linear programming and optimization.

After the feedback session, the four trainees met again and, making full use of the piloting activity, further developed the joint report to the training class, structuring it independently and with additional personal comments and remarks, in the form of a *Unità di Apprendimento* (Learning Unit). This is a teaching methodology recently introduced by the Italian school reform. The Learning Units were used by the four trainees as a part of their work to be evaluated during the exam at the end of the training course.

Third piloting

by Catherine Taveau*

Presentation of the class

The class includes 22 pupils, 14 and 15 years old. It is composed of 11 girls, 11 boys. It is the fourth year of secondary school in France.

Skills implemented and aims

- To intelligently organize calculations about numerical magnitudes (exact or approximate calculations). To use wisely a calculator, spreadsheet and graphics package.
- To invest knowledge about proportionality using a real-life situation where the proportional model may be questioned.
- To represent numerical data with a bar chart; in a coordinate system: choice of coordinates, of scales, of units, ways of representing the above numerical data.
- To use a specific word list: abscissa, ordinate, proportional ratio, chart …
- To propose a situation both mathematically relevant and familiar to pupils, to motivate the use of language and the use of algebra, particularly linear functions, linear equations and inequalities.
- To introduce pupils to linear functions through the study and the writing of literal formulas: status and roles of the letter, of the equals sign, of a literal expression.
- To work in a group: to listen / exchange / share/ produce.

Scenario

Five telephone *FIXED PRICE* contracts: 2 hours and sms

Company 1: *36 €, 100 sms given and then 0,07 € per sms*

Company 2: *29 €, and then 0,12 € per sms*

Company 3: *36,5 €, and then 0,10 € per sms*

Company 4: *21,85 €, and then 0,09 € per sms*

Company 5: *19,90 €, and then 0,09 € per sms*

Three users:

David: 1 *hour and 600 sms*

Marie: *2 hours and 200 sms*

Simon: *1 hour 30 minutes and 60 sms*

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* Institut Universitaire de Formation des Maîtres – IUFM of Paris, France.

Five groups:

The class is divided into five groups of four or five pupils, mixed ability, squaring with the five telecommunications companies. Each group has a hardback company file with the following documents:

- A short advertising slogan describing the fixed-price contract proposed by the telecommunications company.
- Examples of charts and graphs.
- Extracts of individual searches made by pupils two weeks ago.
- Graph-paper sheets.
- A transparent graph paper sheet.

Development in class

The sequence took place on Friday for 2 hours and on Monday for 1 hour:

Five groups are formed by the teacher.

On Friday: The teacher presents the theme and the title of the work “Telephone fixed price contract study”.

Immediately some pupils propose some telecommunications companies and their usual prices.

The teacher gives the instructions orally:

Firstly each group has to compare the telephone bills of the three users referring to the telecommunication company represented by the group. Results will be submitted with a bar chart on the graph paper transparency.

The teacher then writes the instructions on the board.

This first phase lasts for about 35 minutes. The first discussions are about the meaning of the verb «to compare», use of graph paper, choice of magnitudes, of units and graphs. Systematically all the pupils use their calculators. The meaning of operations and results is discussed, particularly the use of approximate calculus, and for some groups proportional processing, or not, to calculate the price of David’s and Simon’s telephone usage.

Secondly two pupils of every group present their prices and their outputs on a graph paper transparency, with an overhead projector. At the end of each report, a discussion begins between the group and the class. This second phase lasts 15 minutes.

Thirdly every group ranks the five telecommunications companies, from the cheapest company to the most expensive one, for each user. The results will be presented in a coordinate system. The X-axis will show the number of sms sent and the Y-axis the amount of the telephone bills in €. Four copies of the coordinate system are given to each group. Only one of these four copies is required at the end of the activity.

Exchanges and work are organized more effectively. Sharing results is necessary within every group to carry out the instructions. This activity lasts 20 minutes.

Photo 12. Pupils are working in groups

On Monday. During forty minutes, within each group, pupils tackle many questions which arouse oppositions or agreement.

- How to distribute calculus better, among them? Is it possible to round up the results? Is it necessary to take into account the telephone usage or only the sms?
- What type of graph to use? In what direction to take the co-ordinate system given by the teacher? Must points placed on the graph sheet be joined or not? What unit on X-axis? On Y-axis?
- How to distinguish, in a same co-ordinate system, telephone companies from users? Whether to use colours, codes, captions, or not?

Outcome

In each group, mathematical work was organized in two ways: the fifteen calculations and the graphic representations. Distribution of tasks and time management were specific to each group.

At the close of these two sessions, many pupils expressed their satisfaction describing the work as a "new and interesting subject, not only about mathematics, far off square roots", "more help between them".

The various graphs and work produced by the five groups showed an important mathematical reflection and good output. However, the mathematical thought processes used by pupils are naturally more numerical than algebraic. So, to become acquainted with linear functions a further sequence is necessary. Then, algebraic language, in particular, will be developed making it necessary and useful to produce and handle literal formulas. So, the next stage of this initiative has to lead pupils to identify the two parts of an algebraic expression – "procedural" and "structural".

Olivier Arrouch, class teacher.

Conclusion

by Annette Jäpelt

The subject has been chosen, because it is a major part of pupils' every day life. The pupils are therefore motivated for 'problem-solving' and can be able to transfer mathematical skills more easily than when taught mathematics only through text-books.

The comparison of different tariff plans available for mobile phone calls is a natural matter of discussion even for pupils in lower secondary schools. While comparing, pupils rarely support their opinions by reasoning that could, even unintentionally, be related to the graphic description of the tariff plans they are considering. They hardly understand that for any correct comparison, several mathematical notions are necessary. Whereas some of them are already available, others still need to be introduced.

The general scheme at the two first piloting institutions is seen in the introduction. In Paris the piloting was a part of this, the piloting in the classroom.

The comparison will be with the two first piloting institutions, when only the teacher trainer and the teacher trainees are involved and include Paris, when the pupils are involved.

The common frame for the piloting

The training session

- Presentation of the proposal for the teacher trainees
- Teacher trainees get or find themselves several tariff plans
- They are, in groups, asked to make a lesson plan for pupils
- In these groups they discusses the best way to introduce and carry through the lesson for the pupils
- These plans are presented in front of the other groups
- Some trainees are selected to the piloting in the classroom

The session in the classroom

- Introduction
- Each group is given several tariff plans for mobile telephones calls
- The groups are asked to compare different plans
- In this comparison are asked for calculations and visualisation in form of a bar chart or the graph for a linear function
- Presentation in front of the other groups

Differences in the piloting

- The mathematics notions involved: in Skårup the notion of variable was used in the treatment and the pupils were drawing a bar chart by hand.
- In the two other piloting institutions the subject was an introduction or consolidation of mathematical concepts as function, graph of a function, direct proportionality, equation of a straight line and the use spreadsheet in the treatment.
- The use of spreadsheet is a very good idea you can change the variables and quickly see what happens, which is useful here with the great amount of variables.
- In the second piloting institution the basic statistical concepts were introduced too.

Concluding remarks

The subject contains a variety of possibilities for using mathematical skills in problem-solving. It is possible to choose how complicated the problem-solving to be, by deciding how many variables to be introduced.

If the competences of problem treating and modelling should be really developed it is important to let the pupils evolve their strategies themselves, without interfering more that absolutely necessary.

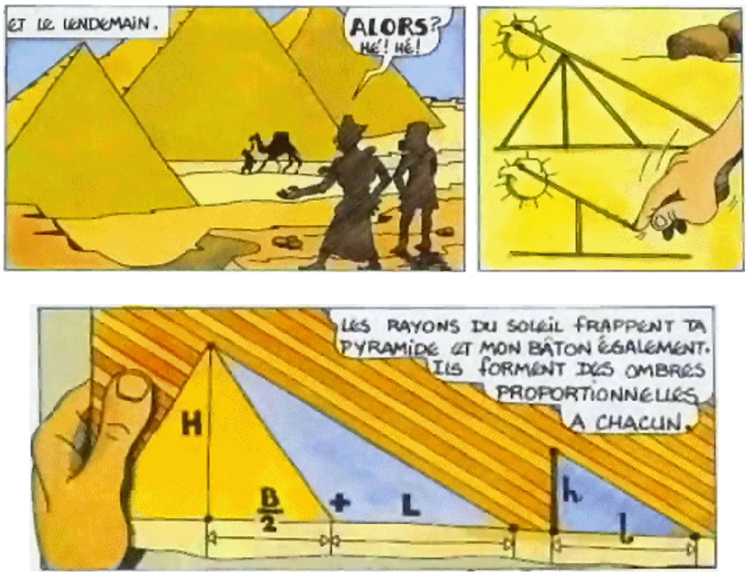
INTRODUCTION TO PROPORTIONALITY IN GEOMETRY

by Yves Alvez[*], Jean-François Chesné[*] and Marie-Hélène Le Yaouanq[*]

INTRODUCTION

This theme combines three fundamental aspects of the teaching of mathematics in lower secondary school. The first one is defined in terms of its mathematical content: it involves proportionality, which constitutes one of the cores of pupils' knowledge. The second one involves a didactic approach: combining a geometrical frame with a numerical context. The third one deals with the use of resources and the integration of new technologies. The combination of these three aspects forms an integral part of the overall scheme of training and is relevant to work with trainees on teaching practices in the classroom.

In the first place, this initiative was piloted by the two following partners: the IUFM of Créteil and the Skårup Seminarium; the external experiment was piloted by the University of Bari, in two grade 7 classes from lower secondary school (pupils aged 12-13).



The legend about the measurement of the height of the Egyptian pyramid[2]

* Institut Universitaire de Formation des Maîtres – IUFM of Créteil, France.

2 Source: http://irem.univ-poitiers.fr/irem/ressourc/histoire_math_college/4ieme/thales/doc_peda/doc_peda.htm.

Main piloting

by Yves Alvez, Jean-François Chesné and Marie-Hélène Le Yaouanq

PRESENTATION OF THE INITIATIVE IN THE IUFM OF CRETEIL

This initiative is an experiment carried out for the first time within the IUFM. From the whole range of training initiatives offered to maths trainee teachers in initial training, we have chosen for the project LOSSTT-IN-MATH the introduction to proportionality in geometry. This initiative demonstrates the trainers' will to set up a training scheme which combines several modules while at the same time taking into account its future application of course for the project LOSSTT-IN-MATH, but also as a training tool within the IUFM itself.

Depending on the year, between 50 and 80 maths trainee teachers for lower and higher secondary levels (PLC2's) go through the IUFM training in Créteil. It includes a 51 hour-long module of practices in the class of mathematics (module A). This module aims at accompanying the trainee teacher, in liaison with the educational adviser tutor, in his discovery of the teaching profession and at facilitating the build-up of his professional experience by providing him with teaching aids as well as pedagogic and didactic elements of reflection: official curriculum, the working out of modes of progression, preparation of sequences and sessions, evaluations, awareness of pupils' diversity, mathematical contents, specific work in algebra or in geometry.

It also includes a module in geometry which aims at consolidating the foundation of acquisitions by trainee teachers, a foundation which will enable them to set up, control and manages pupils' activities, on paper and/or on screen. The first three sessions are devoted to handling computer software programs in dynamic geometry (Geoplan-Geospace, Cabri); each trainee teacher is thus given the opportunity to construct figures, to make them evolve in a dynamic way, and to operate the specific software functionalities (transformation, search for locations, verification of properties, study of functions derived from a geometrical situation etc). The following two sessions allow the return to fundamental configurations of elementary geometry through tutorials, and the opening of ways of designing and setting-up activities of geometry in the classroom. The last two optional sessions, can provide opportunities to discover (or re-discover) in an elementary fashion three geometrical transformations: perspective, similarity and inversion.

As in any training initiative, the description and reflection involved will be carried out with a dual chronological approach: that of trainers towards trainees, followed by that of trainers interested in trainees' practices and their effects upon pupils.

We will therefore specify our objectives and prior expectations concerning trainees, and then we will introduce the training initiative as it has been carried out this year, i.e. its development from beginning to end. This will be followed with a posteriori analysis, still at two levels, that of the session led by a trainee in the classroom and that more global of the initiative in its entirety. Finally, we will present some perspectives open to us as trainers at the IUFM of Créteil and as members of the project LOSSTT-IN-MATH.

A PRIORI ANALYSIS

The modes of the training initiative are worked out with work on practices in mind and not just making a speech and with the will to act upon both the cognitive and mediating components of the teaching profession.

To be more precise, our objectives in this initiative are:

- To make IT technologies available to trainees as learning tools, in particular to make them aware of the benefit of using IT software in dynamic geometry as a special tool for a conjecture and for the discovery of the universality of a property.
- To make trainees work on the design of a pupil's worksheet where the instructions are not limited to the technical aspects of handling the software.
- To make the trainees work individually on a detailed and real scenario. What are the pupils responsible for and what is the teacher responsible for? How to break the session down? What are the difficulties foreseen? Where to fit in the synthesis stages? Which institutionalisation (i.e. results and properties pupils have to get as knowledge according to curriculum)?
- To make the trainees reflect collectively on their individual proposals in order to make them justify their choices.
- To make the trainees deliver efficiently a session on which they will have worked collectively in order to make them realize that it is possible, that "it works", to give them confidence and to encourage them to repeat this initiative (and other novel situations).
- To examine how a novice teacher can make good use of IT resources to facilitate some learning processes for the pupils and establish a comparison with potential activities for pupils during a paper/pen activity.
- To develop a pertinent use of IT in secondary education in accordance with the curricula.

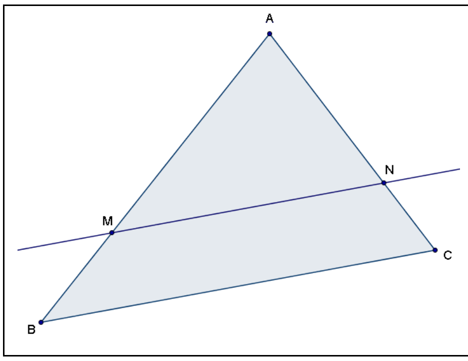
DEVELOPMENT

The training initiative takes place in 4 stages:

- During an IT module (making good use of software programs and instructions)
- During a module A (work on scenarios)
- During an effective delivery (sessions in the classroom)
- Again in a module A (synthesis of experiences)

1st stage

At the end of the first IT module devoted only to handling dynamic geometry software programs, the trainers ask the trainees to prepare an activity for 3rd year pupils aimed at introducing proportionality in geometry. This preparation is done outside the training modules and consists in creating a scenario and designing a pupil's worksheet. The trainees have the choice of two themes – introduction of cosine or introduction of proportionality in triangle, called in France "Thales' theorem" – and two software programmes – Cabri or Geoplan-Geospace. The work is to be sent to the trainers of module A (classroom practices). The IT module is led by two trainers for a group of about fifteen trainees. As for module A it is run by two trainers for a group of between 20 and 25 trainees.



$$\frac{AM}{AB} = \frac{AN}{AC} = \frac{MN}{BC}$$

Picture 7. Thales' theorem

According to the French official syllabus in force for 3rd year pupils in 2004-2005[3]:

Contents	Skills	Examples of Activities, Comments
Triangles determined by two parallel lines cutting two secant lines (see Picture 7).	To know and use proportionality of lengths for the sides of the two triangles determined by two parallel lines cutting two intersecting lines. In a triangle ABC, where M is a point on side [AB][4] and N a point on side [AC], if (MN) is parallel to (BC) then AM/AB = AN/AC = MN/BC	The equality of the three ratios will be admitted after potential studies in specific cases. Of course it also extends to the case where M and N belong respectively to [AB) and [AC) but we will not examine the case where [AM) and [AB) and [AN) and [AC) are opposite. Thales' theorem in all its generality and reciprocity will be studied in the 4th year.

Table 4. Excerpt from the French syllabus

[3] In the new 3rd year syllabus (in force from September 2007), the terms used to define the contents and skills are identical. The corresponding comments however will be replaced with: "***The equality of the three ratios is admitted after prior study in specific cases of ratios***. *It extends to the case where M and N are respectively on [AB) and [AC). The case where points M and B are not on the same side of A is not studied. Thales' Theorem in all its generality and reciprocity will be studied in the 4th year*"

[4] In France the notation [AB] means segment; [AB) means ray; (AB) means straight line

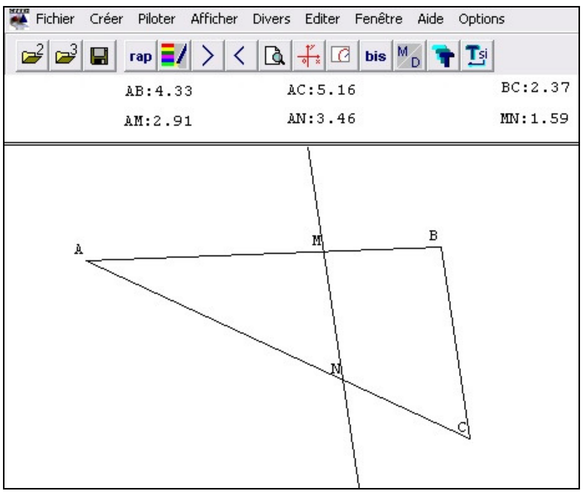
2nd stage

The trainers involved with module A have received the trainees' productions, have read them and annotated them. Most of the trainees have chosen the introduction to Thales' theorem. The session is made of several periods.

It begins with the trainers making general comments mainly on two recurring observations. On the pupils' worksheets, the trainees have not emphasized proportionality. Nearly every worksheet though contains the term "proportionality" in its title, the discovery of which being precisely the main issue of the session! In spite of this, the equalities of ratios are introduced very quickly without the possibility of demonstrating the proportionality of the lengths of the sides of the triangles. The scenarios are on the whole very basic, the different parts are only briefly described, the teacher's part is hardly noticeable. Moreover the formulation of the final conjecture does not stand out from the enunciation of a property.

Afterwards the trainees work in groups of 4 organized by the trainers according to their productions and also to their involvement in the previous modules. (The trainers in so doing endeavour to create dynamic groups, with complementary personalities). Their task is to produce collectively a scenario and a pupil's worksheet that will form the basis for a video projector presentation in the third part of the session.

One of the trainees will therefore then present to all the others the scenario re-worked within his group. Questions are then being asked by the other trainees forcing the presenter and his group to justify their choices, and alternatives appear. The trainers then give a synthesis of the session, bringing in their own comments. It is finally agreed to adopt one common scenario close to that presented. One of the trainees is responsible for drafting and finalizing the documents, scenario and pupil's worksheet (see Appendix A on page 137). The aim of the session is to create a dynamic figure, so that pupils are able to formulate a conjecture.



Picture 8. An example of the expected figure

3rd stage

An effective delivery in all the classes taught by the trainees cannot take place as they do not all have a 3rd year to teach. The filmed session takes place in the volunteer trainee's classroom, without any institutional evaluation. During a previous meeting, immediately before the session, the trainee has introduced his class and presented his project to one of the trainers. In the same way he will after the session make some *on the spot* comments.

4th stage

It is the return stage to module A. The trainee who delivered the session verbally shares with the others his feelings about being filmed and gives a short a posteriori analysis of the session. The trainers complete this analysis. The other trainees intervene to ask questions or to add to what has been said when they too have delivered the session in the classroom. No work however is being done with the trainees using a presentation of the video or an extract from it. There are two main reasons for this: firstly the structure of module A has been worked out before the start of the training and is very complex. Now the type of work undertaken on the introduction of proportionality in geometry and some of the modes of this specific training initiative were integrated during the course of the year. It was therefore difficult, bearing in mind the time constraints, to add a new element in the training. Secondly, video work is still underused in the training of maths PLC2's at the IUFM of Créteil, and neither on the trainers' side nor the trainees' side did anybody feel quite ready to introduce this new tool as an integral component of training.

THE SESSION IN THE CLASSROOM [*this stage was video recorded*]

Presentation of the context

The IT session was carried out with a 3rd year class from Edouard Herriot college, situated in Livry-Gargan, in the Eastern suburb of Paris. The class is made up of 23 pupils, sub-divided into 2 groups, one of 11 pupils, with whom a similar session has already taken place, and one of 12 pupils about to be observed; one pupil per computer.

The teacher is a trainee teacher in initial training at the IUFM of Créteil. According to him, managing the class does not involve any specific problem.

A rigorous follow-up of the pupils' work has been put into place in the classroom on a daily basis and oral participation is taken into account during sessions.

The session is devoted to the discovery of Thales' theorem at 3rd year level. Geoplan has already been used by the teacher on screen and on the blackboard; the notion of pull-down menu has been introduced to the pupils on this occasion. To the teacher's knowledge the pupils have never used Geoplan nor come into the computer room during the course of the year or in previous years, neither for maths nor for any other subject. The pupils are therefore discovering a new workspace.

The questions the teacher may a priori ask himself are the following:

- How will the pupils react to the software program? Will they know how to use the pull down menu? Will they for example know how to find intersections of straight lines?

- Which questions will they ask regarding the functionalities of the software program? (The teacher wonders in particular if the pupils will try to find out whether the calculation of ratios performed on the calculator could have been made with the help of the software program).
- How will the pupils manage with their worksheet?

With regard to the conjecture, the teacher hopes that the pupils will tell him that the completed tables look like proportionality ones and that this statement will be derived from various examples. The theme of proportionality has been dealt with during the previous week, through 2nd year revisions in a numerical or graphical frame: being able to recognize a situation of proportionality (in a table or in a graph), calculate a fourth proportional.

Approximately half an hour later the teacher plans to introduce the synthesis stage and to then have the pupils write in their exercise books, leaving them either in front of the screens or splitting them out at tables in the centre of the room. The beginning of the lesson is written at the back of the blackboard. He plans also to give them a generic drawing to stick on.

Development of the session

1st period (16 min)

There is one computer per pupil. The software program Geoplan is opened at each workstation. The teacher quickly hands out the pupils' worksheets. The pupils get to work quickly and make a reasonably easy start on the drawings. They have a marked tendency to watch their neighbours' screen and to ask about each other's respective progress. The teacher circulates in the room, from station to station.

The first difficulty arises when creating point N. Afterwards a second hesitation occurs over the number of decimals to be displayed for the lengths of the segments.

Finally the pupils get the display of the required six lengths, and then the teacher asks them to fill in a second table. They then undo the initial triangle in order to obtain new measurements.

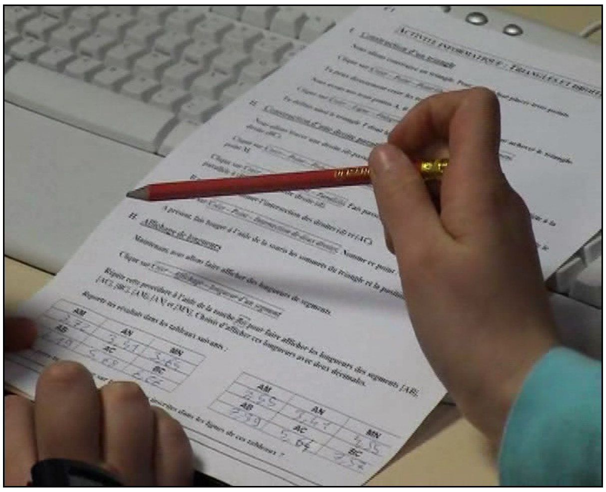
2nd period (17 min)

At this stage, the pupils remain puzzled, firstly as to the nature of the task (what does to conjecture mean?) and then on the task itself (what can they possibly conjecture?).

Then the teacher tries to put them on the right track by mentioning both the tables and what has been covered in class the previous week. The pupils begin by quoting Pythagoras' theorem (we are doing geometry!), one of them mentions proportionality.

Then the pupils take their calculators to determine a possible coefficient of proportionality. But faced with the different results displayed on their calculators, they hesitate and find it difficult to formulate a conjecture in writing. They go back for more calculations on the first table.

The teacher approaches a pupil, and goes over the first calculations with him and then asks him to move on to the second table: The pupil seems a lot more satisfied with the quotients displayed since each of the 3 decimal figures starts with 1,70.



Picture 9. An example of pupils' tables

The teacher goes back to the pupil, modifies the number of digits in the decimal part of the measurements displayed on the screen and asks him to take into account the new measurements and to perform new calculations.

A short discussion follows regarding the nature of the charts (proportionality or not), and the effect of the precision of the measurements.

The teacher asks the pupils to take their exercise books.

***3rd period* (*15 min*)**

The pupils organize themselves quickly, to be able to see the teacher, the chart, and to write at a table. (Some pupils sit at tables grouped in the centre of the room; others stay where they are turning round as necessary).

The teacher begins by making a verbal statement "It can be said that the lengths of the sides of the small triangle AMN appear proportional to the lengths of the sides of the big triangle ABC".

Then he turns the board over, on which the enunciated sentence was already written and asks the pupils to copy it down on their exercise books.

Then he hands out a figure similar to those already studied by them, puts one on the board and certifies that "in such a configuration the chart is a proportionality one".

The teacher then writes the following property (admitted): in a triangle ABC, if M is a point on side [AB], if N is a point on side [AC] and if the straight lines (MN) and (BC) are parallel, then the lengths of the sides of the triangle AMN are proportional to the lengths of the sides of the triangle ABC. The pupils in turn write down the property in their exercise books.

A POSTERIORI ANALYSIS OF THE SESSION IN THE CLASSROOM

The pupils are well managed; their involvement is noticeable and sustained.

The teacher has not included any intermediate synthesis stage: his interventions are always personalized; the difficulties encountered by the students are therefore repeatedly dealt with.

During the course of the first period (construction of the figure and display of lengths measurements), the use of the software program appears to have had a significant driving effect on pupils' participation in the tasks required of them. The structure of Geoplan-Geospace has demonstrated the importance of designating geometrical objects: several pupils were thus confused when small letters used as capital letters were being rejected. Generally speaking, the necessity of defining the objects to be created has forced the pupils to pay more attention to the vocabulary used in geometry. The display of length measurements has also led the pupils into asking questions regarding the number of digits in the decimal part. However, there was no question from them regarding the unit of length used, nor on the choice of the number of digits to be displayed.

The transition from observation to conjecture was not spontaneously made by the pupils. Even after several verbal reminders from the teacher, the pupils did not mention any situation of proportionality. Only when the teacher referred insistently to other charts used the previous week, did one pupil mention proportionality. The "Topaze effect" is here clearly in evidence.

At this stage in the session, it is to be noted that the teacher has deliberately adopted a strategy consisting in the substitution of work on measurement charts with a calculator for work based on the observation of direct display of ratios by Geoplan-Geospace. The teacher gave two reasons to justify his choice: firstly he feared difficulties related to the pupils' handling of the software, for whom this was the first session on Geoplan-Geospace, secondly, he took into account the preparation of the session done in training where the risk of obscuring the proportionality when working directly on ratios had been mentioned.

But the difficulties encountered by the pupils with their calculations, which resulted directly from his choice, had not been foreseen by the teacher. Indeed, the results displayed on the calculators are not equal. It is therefore natural that the pupils tend not to conjecture a situation of proportionality. The teacher handles "this didactic incident" by proposing to the pupils to increase the number of decimals in the display of the lengths measurements.

One would have thought when reading the pupil's worksheet that the second table aimed at consolidating a belief derived from work on the first one: but the order in which the pupils' activities actually run – filling in the tables, then calculations of the ratios for both tables, as opposed to work on one table, conjecture, then work on the second one – has come to contradict the original plan, and has therefore significantly weakened the experimental process.

Finally one can also wonder how the pupils really perceive proportionality, for the observed session can lead us to believe that it is the didactical representation often used in the classroom – the chart – that triggers the recognition of a mathematical situation. In other words, it obviously appears that here the notion of proportionality is not available, and that is the usual representation, supported by the teachers' speech, which brings up the conjecture aimed at.

The last period of the session, the transition from conjecture to property, is rather moderately managed by the teacher: no work on proof, not even a speech on this very transition, on the reasons behind his choice. One can wonder about the effects generated by this situation on the pupils in terms of learning: Don't the pupils in fact remain with geometry of perception (one sees that … it can be said that …)? Which status do pupils really attach to the property aimed at during the session: universal or relative to their drawing? Is it really always true or in some cases only, or "nearly true" sometimes?

By the end of the session, a number of variations can be envisaged:

- Two pupils per workstation.
- Make the pupils work on a drawing (with show of hands) before making them work on computers (lines, observations and first conjectures).
- Have the ratios displayed by Geoplan-Geospace.
- Set in the first chart the length of side [AB] of a triangle and the position of M on [AB], then move M on [AB].
- Make the pupils start from situations corresponding to ratios 1/2 or 1/4.

A POSTERIORI ANALYSIS OF THE TRAINING INITIATIVE

This whole experiment is a source for new questioning, regarding the subject dealt with in the classroom on one hand and teacher training on the other hand.

Teachers in initial training (and the others too) are forced to follow official instructions. The analysis of the session carried out in the classroom and in particular the teacher's choice to drop the display of ratios for the use of tables and a calculator raises the following question: Does the wording of Thales' theorem as it appears currently in the syllabuses encourage work to be done on the ratios or on the aspect of proportionality in geometry, as will be the case with similar triangles in the 5^{th} year? The answer is not an institutional one since this double aspect appears in fact quite explicitly in any of the three columns, contents/skills/comments of the syllabus. It remains therefore for the trainer to make up his mind in view of the "good practices" aimed at by trainee teachers. Choosing to have the pupils work directly on the ratios has certainly the advantage of enabling them to arrive more easily at a conjecture, the equality of the ratios, but incurs the significant risk of obscuring the proportionality of the lengths. On the other hand, trying to make the pupils conjecture a situation of proportionality may seem more meaningful, but more difficult to achieve, with the risk of a diminished involvement from the pupils. It has to be stressed that in both cases, the work is carried out on the measurements, and not on the magnitudes.

In order to take into account both the double aspect mentioned above and the a posteriori analysis of the session, it seems therefore reasonable, and reading the new syllabuses seems to suggest it, to propose the following steps to trainee teachers:

- To start with pen/paper work on simple cases.
- To have the pupils conjecturing these specific cases in terms of "...times more…" "… times less …"
- To translate this conjecture in the form of an equality of ratios.
- To resort to computer software in dynamic geometry to reinforce this conjecture by using the display of ratios.
- To enunciate the property aimed at, in the double version proposed by the syllabus, proportional lengths and equality of ratios.

CONCLUSION

Besides the training objectives defined in the first part of this presentation, work in training, based on the construction of a scenario of one identical session for all the trainees has been retained *a priori* for two main reasons:

- It facilitates module work (before and after)
- It encourages the study of the effects of teacher's practices on pupils' activities from one single prescribed task.
- The reflection carried out during and after this work allows us to think that this effective training has indeed complied with the initial objectives and seems to confirm the relevance of this type of initiative at several levels:
- The integration of TICE (IT) in trainees' practices
- The importance of the production of written scenarios and of their application
- The diversity and depth of the exchanges between trainees/trainees and trainers/trainees
- The emergence of a typical scenario, the validity of which, yet to be tested, would be justified by a good compromise between a high potential degree of appropriate use by the trainees and a satisfactory degree of effectiveness in the classroom (the first point being necessary for the second one to happen, but in no way sufficient).

However, the video work carried out with one trainee, does not guarantee in any way an impact on the other trainees. The problem of the evaluation of the knowledge acquired by the pupils in the filmed trainee teacher's classroom and also that of the evaluation of the effect of such an initiative on the practices of the other trainees remain therefore on hold.

SUGGESTED READING

Alvez, Y., Le Yaouanq, M.-H., Carême, A., Chareyre, B., Cleirec, N., Gastin H., Guillemet, D. & Saint-Raymond C. (2005). *Collection Math'x: Seconde, Première S, Terminale S*. Editions Didier France.

Autour de Thalès (1995). Commission Inter IREM Premier Cycle.

Brousseau, G. (1995) Promenade avec THALES, entre la Maternelle et l'Université. In *Autour de Thalès*, IREM de Lyon Villeurbanne.

Hersant, M. (2005). *La proportionnalité dans l'enseignement obligatoire en France, d'hier à aujourd'hui*. Repères IREM N° 59, TOPIQUES éditions Metz

Robert, A. (2005). Quelles différences y a-t-il…? Exemples d'analyses didactiques d'exercices ou d'activités élèves (collège ou lycée). *Bulletin APMEP 457*, 226-238.

Web links[5]

French Curricula

[http://eduscol.education.fr/D0048/LLPPRC01.htm] and

[http://www.cndp.fr/secondaire/mathematiques/]

A.I.D. (*Association pour l'Innovation Didactique*) *C.R.E.E.M.* (*Centre de Recherche et d'Expérimentation pour l'Enseignement des Mathématiques*)

[http://www.aid-creem.org/telechargement.html]

C.R.E.E.M. was a specialised centre within CNAM (Conservatoire National des Arts et Metiers) created in 1972. For more than 10 years C.R.E.E.M. was a privileged partner of the Ministry of National Education in the management of Lycees and Colleges, and later in that responsible for developing new technologies (DITEN, and then DISTNB). On 26th February 2003, CREEM closed its doors for good. It has been replaced with the Association for didactical innovations

Geoplan/Geospace

[http://www.crdp-reims.fr/Ressources/lib/Titres-reseau.htm?produits/pdt118.htm]

Geoplan-Geoplace is windows-based mathematics software for use from primary school to university. Geoplan-Geospace is software for mathematical constructions that allows dynamic and interactive representations. It allows the user to define and manipulate numerical and geometrical objects in a plane or in space (points, straight lines, circles, spheres, solids, convex polyhedron, numbers, transformations, loci, curves, vectors, numerical functions, numerical sequences, prototypes, etc.). These creations and manipulations can be automated by adding new commands.

Cabri [http://www.cabri.com/v2/pages/fr/index.php]

Cabri II Plus is a tool for the realisation of geometrical constructions, just as you would on a piece of paper with a pencil, ruler, compass, and rubber. The software brings a new dimension to these constructions: the figure and its elements can be freely manipulated by the user and the construction is immediately updated. The constructions can be integrated into documents (under Mac, Windows) or published on the Internet (Cabri.Java).

[5] Active on December 2006.

Second piloting

by Annette Jäpelt*

AIMS

For trainees:

- To develop a lesson plan involving a dynamic geometry programme.
- To learn how to use information technology in teaching.

For pupils:

- To learn how to use a software programme in dynamic geometry
- To derive the connection between proportionality and similar triangles using a dynamic geometry programme.

PROPOSAL

First the trainee teachers must learn to use the dynamic geometry programme. We use the programme Geometer, a Danish version of Geometer Sketchpad. Through exercises the students become familiar with the most common facilities of the programme within classic geometry.

After that, we look at the idea on which we want to focus: the connection between proportionality and similar triangles.

Definition: *Two figures are similar, when one of them is an enlargement of the other.*

The students know how to use the connection, but have not yet explored in depth the theorems that specifically apply to this idea.

After the lesson with the pupils we will take a closer look at these theorems, in order to prove and use them. Partly the theorem of multiplication around a point, and partly the idea of similar triangles and their properties:

Triangle ABC has equal corresponding angles to triangle \`A\`B\`C\`.

Triangle ABC and triangle A\`B\`C\` have corresponding angles that are equal.

The **ratios** between the corresponding sides of the two triangles are equal, i.e.

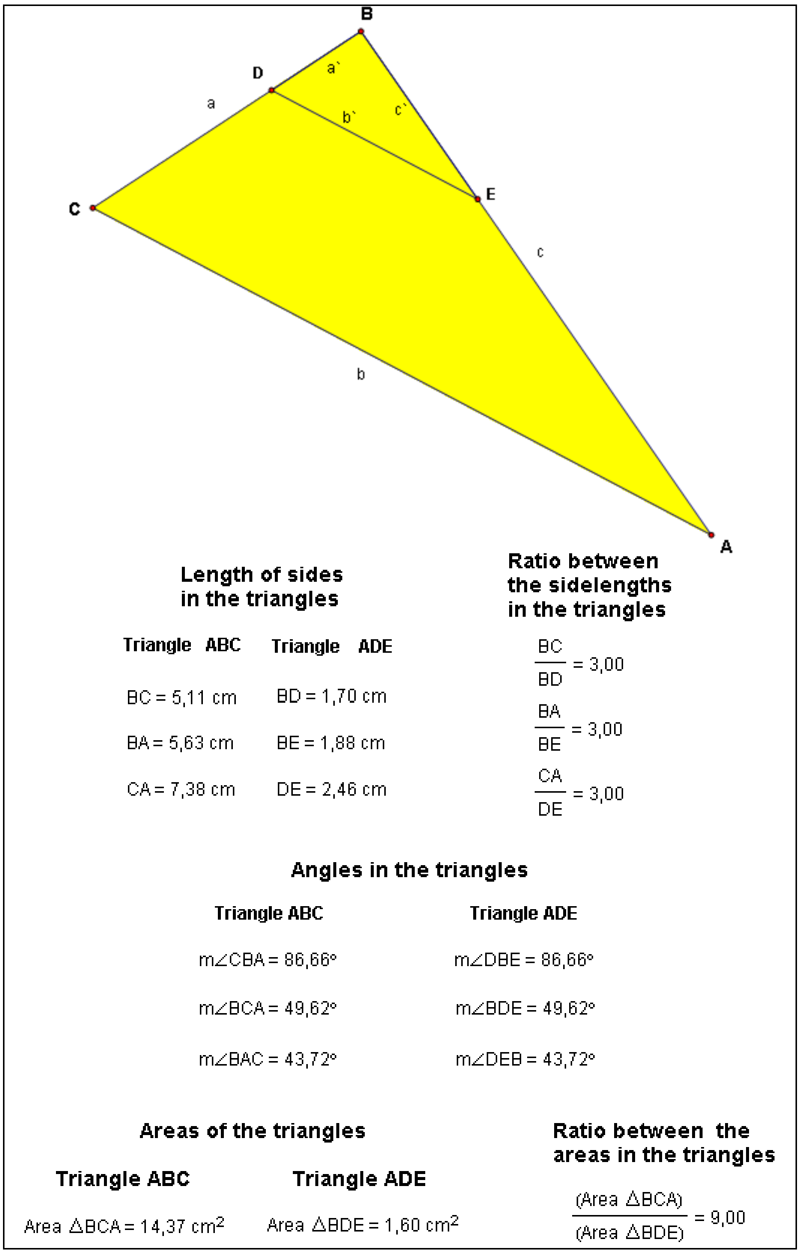
$$\mathbf{a'/a = b'/b = c'/c}$$

Once the trainee teachers have become familiar with the Geometer program and the properties of similar triangles, their task is to then prepare a lesson for a (school) class. The aim of the task is to introduce them to the programme and to the properties of similar triangles.

The course which ends here is followed by an evaluation.

* Skårup Seminarium, Denmark.

Later the trainees will have a study week in mathematics, during which the teacher trainer have planned two days of surveying about proportionality in geometry.



Picture 10. Illustrating proportionality

INSTRUCTIONS FOR THE TRAINEE TEACHERS

There are about 25 trainee teachers at Skårup Seminarium. There is a wide age range and the gender distribution is about fifty/fifty.

Prior to starting this phase, the trainees have been practicing with the Geometer program as an introduction to its more general uses (operations, drawings, measurements and calculations). This was only meant as an exercise before the student teachers start to work on the preparation of the lesson for the pupils.

I gave the trainee teachers the following introduction:

> “Your objectives for the next two weeks are to be able to make an introduction and create a document enabling the pupils to:
>
> - Draw triangles using the Geometer programme
> - Learn about proportionality and similar triangles by means of the dynamic geometry programme.
>
> Trainee teachers work in groups and each group makes a proposal; the whole team then selects the best one.
>
> This task should take one week (four lessons)”.

I have promised the class teacher in which this session will be piloted, to summarize this part. If needed, I will devote one additional lesson to the IT section. I will let the pupils use the mathematical knowledge they have previously acquired to carry out an actual survey, i.e. find the height of a tree using concrete measurements and then make calculations using the properties of similar triangles.

I would also very much like you to consider other real life situations where pupils can put into practice the concept of proportionality.

The trainees work with Geometer in the computer room on the connections with similar triangles. They work on establishing (property 1 and 2 above).

I suggest the trainees to do the following:

- Draw a triangle ABC.
- Enlarge or reduce the triangle by a given scale factor, using the instructions in the software (the result will be that the sides are increased/decreased by a given factor). You now have a new triangle ADE.
- Measure the lengths of the sides of the two triangles ABC and ADE.
- Find the relationship between the similar sides of the two triangles.
- Measure the angles of the two triangles.
- Let the points vary.
- Let the ratio vary.
- Derive the relationship between the sides and angles of similar triangles.
- Measure the area of the two triangles and derive the relationship between the areas.

In the above exercise, we were varying the sides first and then the angles.

In the second method we can also do it the other way round, by first making the angles equal and then looking at the relationships between similar sides. What we do here is to start with parallel sides and then find that the triangles are proportional.

Do the following, using Geometer:

- Draw a triangle ABC.
- Draw a line inside the triangle. This line must be parallel to one of the sides. for example parallel to the side BC. Thus you have a new triangle ADE.
- Measure the lengths of the sides.
- Find the relationship between the similar sides of the two triangles.
- You can then let the points and also the parallel line vary.
- Derive what applies to the two triangles.
- Measure the area of the two triangles and compare the relationship between areas to the relationship between the sides.

Here the angles are identical and you find that the lengths of the sides are in a fixed ratio. If you have any time left you can consider which practical examples could be incorporated.

The teacher trainees work on these points during two lessons.

The teacher trainer has chosen to let the whole group instruct the pupils as it would be good for the trainees both to prepare the lesson and to instruct the pupils.

As most of the pupils must sit alone or in pairs at the computer it will be best to have one of the trainee teachers at each computer.

The pupils have not yet used mathematics programmes so it will be beneficial if each one can have personal instruction.

Furthermore, the trainee teachers will get a good idea of what pupils can understand when they are later going to teach a whole class. I also hope that this will encourage trainee teachers to use IT in their maths lessons when they too later become teachers. A threshold may have been overcome, and they may be less reluctant to use mathematical software when they themselves become teachers.

The trainee teachers work in groups to make proposals for the pupils' lesson. Afterwards the whole group decides how to conduct the lesson.

INSTRUCTIONS FOR THE PUPILS

One lesson is planned. The lesson will take place in the computer room at Skårup Seminarium.

The trainee teachers' instructions for the pupils:

Lesson on Proportionality.

Start the program GEOMETER

Make 3 points; A, B, C

Connect the points to create a triangle

Measure the angles of the triangle

Measure the sides

Multiply by 2 from point A, i.e. you enlarge the lines from point A (2:1)

When the lines AB and AC are enlarged, the new points are renamed. Connect these two points to create a new triangle.

Measure the lengths of the sides and the angles of the new triangle

Can you see any similarities between the two triangles?

Which similarities do you see?

Can you formulate a theorem from your observations: if necessary, draw more triangles to verify your theorem?

*Find the area of the two triangles (½ height * baseline)*[6]

What is the relationship between these two areas?

Some trainees chose to work more independently, but most of them decided to use the printed instructions.

PROGRESS OF THE TRAINEE TEACHERS' LESSON

Two lessons were used on self-test of the exercises as the basis for similarities.

The trainee teachers arrived at significantly different findings during these two lessons. It is an advantage for everybody to use IT, but some happily accept the media and use them while others are not very confident and are very slow; the biggest group however lies between these two extremities.

As before, the following two sessions were devoted to preparing the lesson for the pupils and again the whole group selected the best proposal.

PROGRESS OF THE PUPILS' LESSON

There are twenty pupils from Øster Åby Friskole. They are about 14 years old.

One lesson is planned. This lesson takes place in the computer room at Skårup Seminarium. The pupils had little knowledge of proportionality and had never used mathematical software.

Some of the teacher training students had visited the class once before, during the lesson about mobile phones. The maths teacher of the class was present during the lesson, but only as an observer.

The pupils will work one by one or two by two and there is at least one trainee teacher to instruct them at each computer.

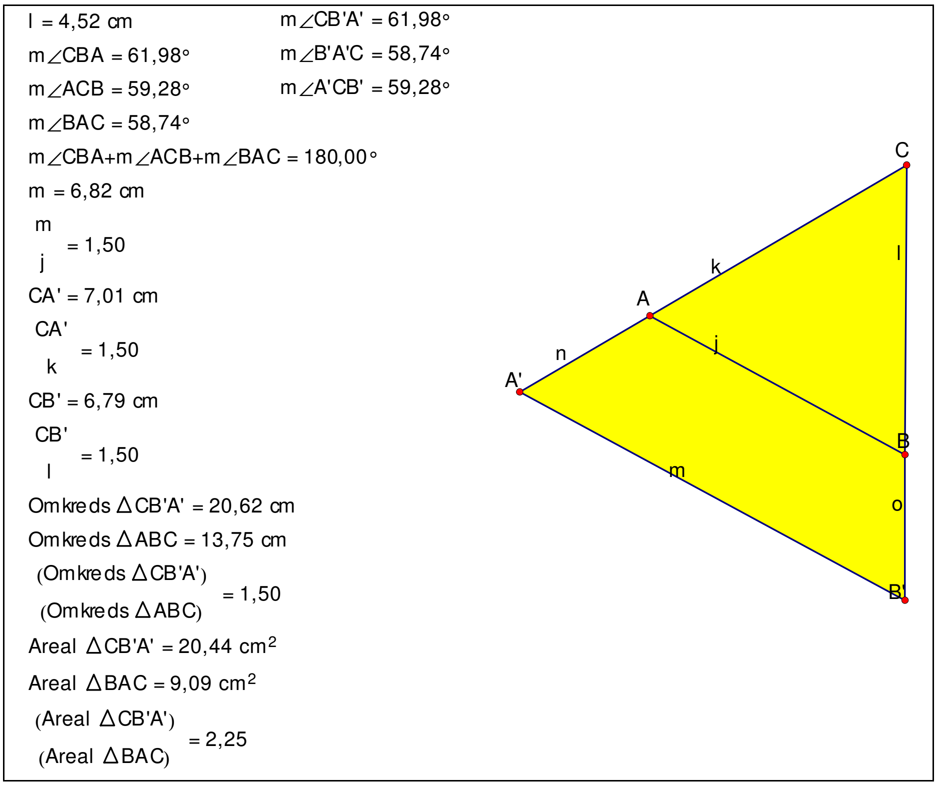
One of the trainee teachers welcomed the class briefly and introduced the lesson. Then the pupils started to work *using* Geometer. The trainee teacher's introduction to the pupils can be seen under "Instruction for the pupils".

Every group reached all the described points. The cooperation between the pupils and the teacher training trainees was good. Generally speaking the pupils were actively

[6] It is possible to measure the area without computing it and I (the teacher trainer) find this much better as this stage, because the focus here is not on how to calculate the area, but on proportionality.

engaged on all points requiring the use of the geometer program and the teacher training students were good at helping them.

Some pupils succeeded in deriving general conclusions but the variation of their products was wide and because the trainees were anxious to get results, they led some pupils to the required conclusion. After the lesson the pupils handed in their work.



Picture 11. An example of one of the pupils' drawings

Remark: As all calculations inside the programme are done directly from the measured values, it's easy to make a direct comparison between the ratios without taking into account the position of the triangles in relation to each other.

EVALUATION

The feedback from the maths teacher of the class was that he believed it was a good lesson.

As basis for a dialogue I have created an evaluation scheme for the trainee teachers. The evaluation scheme is shown below with some answers already inserted.

Task	What was good?	What was bad?
Preparation for the pupils' lesson	*Work two by two; many ideas came up, good to exchange/discuss ideas. Best way of learning.* *Finding out which knowledge is required before the work with IT*	

Pupils' learning	*Interesting to see the pupils' "aha" surprising experiences.* *That they had to tackle the problems themselves.*	*Too little space in the computer room.* *Not enough computers.*
What did you learn from the lesson?	*How you can plan a lesson in which IT is used in a pertinent and effective way.* *Found out that I have to learn more about Geometer.* *Assimilation facilitated by using the programme in mathematics.* *I gained a much better understanding of the programme while using it for something concrete.*	

Table 5. Evaluation scheme for the trainee teachers

COMMENTS

The following session with the trainees is a theoretical connection between proportionality and similar triangles. Normally I would chose to prove the theorems about similar triangles and proportionality without any discovery activity, but in this case the proposal is to first experiment the properties through IT. The discussion will in this case be about the order of the activities: hypothesis and proof first or experimentation through IT. Personally I think we should vary. As for the trainees there is no unambiguous opinion. It seems that the younger ones would prefer the IT way while the others are more divided. An explanation for this may be that young trainees are often more familiar with computers.

It has been very fruitful, both for me as a trainer and for the trainees too, to be so directly involved in the planning of a lesson as well as in its implementation in the classroom.

Trainers are normally involved as advisers rather than participants during the trainees' practical training: personally, thanks to the experience gained from this project, I will endeavour to change this. It has been very relevant to follow the whole process, from the trainees' initial learning to the delivery in the classroom after preparation of the lesson. The trainees' involvement has been considerable and their reflection sustained. The number of students actively engaged increases as they can see the direct relevance of what is being taught. It is highly motivating to be able to put immediately into practice what has been learnt.

I wish that the whole teacher training could be based on a regular interaction between theory and practice, so that maybe every month we were in dialogue about concrete training issues.

Follow up

The trainee teachers used similarity partly through practical measurements.

Examples of this could be: width of a river, experience how a woodman measures the height of trees, the slope of a roof for carpenters, interaction with Physics/Chemistry regarding distances in the universe and the determination of wavelengths, interaction with geography regarding scales and map reading.

Acknowledgement

Thanks to Øster Åby Free School, seventh class and their mathematics teacher, Brian M. Østergård.

Also thanks to my mathematics class 22.4 at Skårup Seminarium for their kindness.

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Web links[7]

The Geometer's Sketchpad [http://www.dynamicgeometry.com/]

Sketchpad is a dynamic construction and exploration tool that enables students to explore and understand mathematics in ways that are simply not possible with traditional tools or with other mathematics software programs. With Sketchpad, students can construct an object and then explore its mathematical properties by dragging the object with the mouse. All mathematical relationships are preserved, allowing students to examine an entire set of similar cases in a matter of seconds, leading them by natural course to generalizations. Sketchpad encourages a process of discovery in which students first visualize and analyse a problem and then make conjectures before attempting a proof.

[7] Active on December 2006.

Third piloting (at the University of Bari, Italy) and conclusion

by Yves Alvez*, Jean-François Chesné* and Marie-Hélène Le Yaouanq*

First impressions

The initiatives carried out have shown a real interest, both from the trainees and from the pupils, in using geometry software, whether they had used this kind of software before or not. They have also revealed a necessity to ensure that teachers are properly trained to take into account the use of IT as a tool in the work they do with their pupils.

The idea was to enable trainees to use ICT in the classroom hoping that, in spite of possible material constraints, their first experience would prove fruitful enough to encourage them to continue. This objective has been achieved by most at Créteil and Skårup.

THE THIRD PILOTING

The theme of proportionality in geometry was tested in the training of trainee teachers at the University of Bari, Italy, by using the training initiatives carried out by the IUFM of Créteil and Skårup College of Education.

Below you will find an account of the experiment carried out by R.I. Ancona and M.A. Giovinazzi, two lower secondary school teachers.

Participating classes

Two year 7/8 classes from lower secondary school (pupils aged 12-13):

- Scuola media statale "E. Fieramosca", Barletta (Ba) composed of 23 pupils.
- Scuola media statale "A. Manzoni", Massafra (Ta) composed of 25 pupils.

Time allocation, instruments and materials

In the planning phase we envisaged at least 4/5 hours of laboratory-based activities (besides the classroom mathematics activity). In the actual implementation only 3 hours were devoted to laboratory-based activities.

Besides standard teaching materials the software "Geogebra" was used in class A and "Cabri II plus" in class B.

A priori analysis of the classes involved in the teaching project and planning of the activity

In class A there is a wide range of abilities; the working environment is collaborative, with a generally active participation by pupils in collective discussions. Pupils have

* Institut Universitaire de Formation des Maîtres – IUFM of Créteil, France.

good computer skills, but no previous experience in using dynamic mathematics software.

In class B the average level is higher (only two or three cases of little cognitive disadvantage can be highlighted). Pupils are expert Cabri users.

In both classes the theme “ratios and proportions” was introduced and dealt with in the weeks preceding the experiment.

The aims of the experiment were as follows:

- monitoring in detail the effects of using an interactive educational software for teaching geometry;
- verifying whether structuring suitable teaching worksheets allows us to monitor the intuitive reasoning stimulated by the use of this software.

Two stages related to the planning of the computer-based activity were envisaged:

- introduction of the Geogebra software (this phase concerned only one of the two classes involved in the project, as the other class was already familiar with Cabri);
- handing-out a practical worksheet related to the “Discovery of Thales’ Theorem”.

The type of questions to be included in the structured scheme varies according to their nature and their aims. We envisaged:

- questions for reflection and testing on software-based constructions;
- questions for reflection on the dynamic actions carried out;
- questions on non-formalised hypothetical and spontaneous deductions;
- check-questions on previous reasoning paths followed by pupils working in pairs.

The worksheet was handed out to pupils working in pairs.

Analysis and Reflections about the experiment

Use of an interactive software and comparison:

We did not record relevant difficulties in neither class concerning the proposed constructions. However their limited knowledge of Geogebra software did not allow pupils to use the software for calculating ratios between sides when they had to fill in a table related to the ratios between the sides of the two triangles. This led them to face the problem of approximate calculus, which in turn led them to find “close” rather than “equal” ratios most of the times (“we notice that the results of the divisions look alike”).

Both classes are used to working in pairs, especially in the computer lab. The “software expert” class showed a greater capacity for synthesis, whereas in the other class there was a clear need to “describe” the slightest observation in great detail.

Discussions were good especially in pairs with a cognitive gap which was neither excessively high nor low.

Dynamic aspects of reasoning and reached conclusions

In the "*expert*" class the use of Cabri supported the construction of reasoning; it actually helped to test intuitive ideas pupils considered valid (for instance, the idea many had and quickly refuted, that "similarity" between triangles entails "unitary ratios between their sides"). In the case of a line cutting the triangle by a point D, though, the only highlighted aspect is a clear recognition of different triangles, without reflections about ratios between sides.

The use of Geogebra enabled pupils to control some geometrical aspects, but the outcome of operations with calculators brought about embarrassment. However the worksheet asked pupils to move point D many times and record the ratio; therefore in some cases the values were "almost completely" equal; moreover the worksheet focused again on ratios thus enacting a further phase of control of previous reasoning.

In the final phase, several pairs tend to check ratios when faced with the problem of another line intersecting the triangle by D.

Reflections on the whole experiment

Certainly this experiment tends to confirm that the use of a dynamic geometry software adds value to the activity, making geometrical objects dynamic and confirming/reflecting/constructing mental images. Nevertheless, one cannot do without the simultaneous use of instruments that permit the checking of the intuitive thoughts that accompany this type of work. We believe that the worksheet was anyway a useful instrument not only to understand all the processes that were going on but also because completing it involved sharing ideas within pairs. What is certainly missing is a further phase of general "sharing" of ideas when designing a "classroom worksheet".

DIFFERENCES

However, some differences have emerged between the different partners regarding:

- the choice of software and related activities (the software interfaces lead to the construction of figures in a certain way, inferring upon reasoning and learning processes);
- the objective of the session in the classroom in terms of mathematical content;
- the conditions required for implementation in the classroom;
- the degree of integration of ICT into the curricula in various countries;
- the importance given to ICT in teacher training;
- how familiar the trainers are with ICT, how they use it both when teaching their pupils and when training trainee teachers.

The assessment of the session in the classroom confirms the benefits of using dynamic geometry software but reveals also that just knowing how to use them well technically is not sufficient.

The pupils have to be taught how to study the results they have arrived at and how to relate them to previously acquired knowledge if they are to draw, by themselves, pertinent conjectures from their observations.

OVERALL ASSESSMENT OF THE TRAINING INITIATIVE WITH TRAINEE TEACHERS

The three training initiatives have highlighted four stages in the work carried out with the trainees:

- use of geometry software;
- design of a worksheet for pupils;
- implementation in the classroom;
- mutualisation and analysis of the experiments carried out.

It would appear that the benefits derived by the trainees from the training initiative are much greater when combining these four stages.

The design of a worksheet for pupils brings about one significant alternative: is it advantageous to draw up a worksheet for pupils common to all trainees, and if so at which stage (before or after the sessions delivered in the classroom)?

The issue in the preparation of any training session is whether it is more sensible to start from a document created by each trainee individually and to collectively develop it during the mutualisation phase, or to suggest a collective construction work towards one common document.

Choosing to create one worksheet common to all trainees before the session in the classroom results in a mutualisation phase easier to manage and places emphasis on the influence of classroom practices upon pupils' activities. However this makes it potentially more difficult for each trainee to use the document individually.

On the other hand, creating individually a worksheet for pupils allows each trainee to prepare the whole of his/her session, thus facilitating their involvement and emphasizing their responsibility in the construction and the development of the session but does not lead them to such a comprehensive analysis; this is even more so in the case of a session involving the use of ICT.

The assumption is that professional training is enriched by the coexistence of all these different types of scenarios.

Appendix A: *Proportionality in geometry* – Pupils' worksheet

ON THE COMPUTER: TRIANGLES AND PARALLELS

Construction of a triangle

We are going to construct a triangle. For that, we need three points.

Click on *Create – Point – Free point – In the plane*

You can create the three points A, B and C directly.

A, B, C are there. We will join them to complete the triangle.

Click on *Create – Line -Polygon – Polygon by vertices*

Now the triangle T has been created. Its vertices are A, B and C.

Construction of a line parallel to one side of the triangle

We are going to draw a line d that passes through M on ray AB, parallel to the line through BC.

Click on *Create – Point – Free point – On a ray*. Select the ray AB and type in M for the point.

Click on *Create – Line – Straight line – Parallel*. Make sure this line goes through M and is parallel to the line through BC. Call this line d.

Now we have to type in a name for the point where the line d and AC intersect.

Click on *Create – Point – Intersection 2 lines*. Call this point N.

Now, click on the vertices of the triangle and on the line d and drag them.

Display of lengths

Now, we want the lengths of some segments to be displayed.

Click on *Create – Display – Length of a segment*

Use the toolbox Bis to repeat this procedure in order to get the lengths of segments AB, AC, BC, AM, AN and MN successively. Get the numbers displayed up to two decimals. Enter your results in the following tables:

AM	AN	MN
AB	AC	BC

AM	AN	MN
AB	AC	BC

What links can you find between the numbers written in these tables?

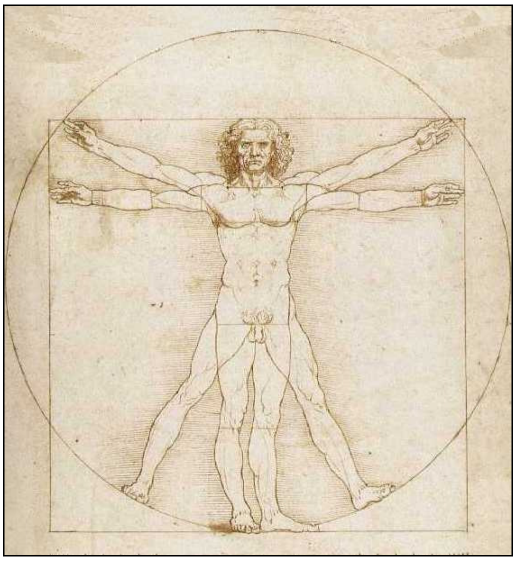
BODY MEASURES

by Brunetto Piochi*

INTRODUCTION

The Body Measures activity is related to measure and to some themes from arithmetic (ratios and proportions) and statistics (mean median, correlation…). Due to how the two partners structured and piloted it, the activity can also be used to suggest an approach to pupils to themes from history of sciences and can as well be used as an introduction to computing by pc and to representing measures on a graph.

Trainees are asked to make some measurements of their bodies (height, weight, length of arms, etc.). Some arithmetical or statistical computations are made using these measurements, with the aid of EXCEL software, while looking for meaningful ratios or correlations, which can also be related to Leonardo da Vinci's hypotheses on the anatomy of human body. Similar activities will be performed with pupils and the results of their piloting will be afterwards discussed with trainees.



Vitruvian man by Leonardo da Vinci

Leonardo studies the proportions of the human body and its commensurability with the perfect geometric forms (the circle and the square). This was scientific analysis that had both cosmological meanings (the correspondence between micro-and macrocosm) and artistic ones (correctly representing the human figure and designing architecture based on the proportions of the human body). In this famous drawing from Venice, Leonardo subjected the "Vitruvian man" to a series of original development.

From the exhibition "La mente di Leonardo" held in Florence on September 2006

* Department of Mathematics, University of Florence, Italy.

Main piloting

by Brunetto Piochi

A GENERAL OVERVIEW OF THE ACTIVITY

Aims

For trainers

- Guiding trainees from theory to practice
- Letting the trainees experience an activity on their own, before proposing it to the pupils
- Providing instructions and feedback

For trainees

- Discussing about measuring and connected didactical arguments
- Knowing about the historical evolution of measuring (in particular length, weight, capacity, …)
- Experiencing measuring with respect to a given unit and working on measures

For secondary school pupils

- Experiencing measuring with respect to a given unit and working on measures
- Knowing about the historical evolution of measuring (in particular length, weight, capacity, …)
- Measuring by international standard units
- Understanding the meaning of "approximation"
- Computing the mean and median of data series
- Representing measures on a graph.

Description of the piloting of the activity

Activities in SSIS (Specialisation School for Secondary Teaching) were carried out with about 30 first year trainees, specialisation Natural Sciences, for getting the teaching qualification for Mathematics and Science in the Lower Secondary School.

Phases and timing:

- Introductory lesson on measure and presentation of Leonardo da Vinci's excerpt (45 min)
- Measuring activity, data processing and discussion (1hr 30')
- Piloting in the classroom (3 hours)
- Final discussion and definite outline of the proposal (45')

SSIS students, after a theoretical introduction on the meaning of measure and its history, were presented a text by Leonardo da Vinci about "Vitruvian man"; among

Leonardo’s statements we chose some that better suited an experimental verification, in particular:

“The length of a man’s outspread arms is equal to his height”

“From the elbow to the tip of the hand will be the fifth part of a man”

“From the bottom of the chin to the top of the head is one eighth of his height”.

Trainees measured one another, then reported obtained measurements on EXCEL worksheets in order to verify whether Leonardo’s hypothesis was correct.

In the discussion that followed the activity students were invited to answer the following questions, of course focusing on the didactical aspects of the activity:

- What competencies are involved in this type of activities? What prerequisites are necessary? What learning is it aimed at?
- What difficulties did you meet in this activity? Do you believe pupils would meet further difficulties? How can we help them overcome these difficulties?
- How much and which statistics is involved in the activity? How can we capture pupils’ attention and focus them on the approximation level?

Later, two trainees, already teaching in a school, carried out piloting in the classroom: this allowed them to work with known classes and insert the activity within the curriculum. The proposal, sketched during the preliminary discussion, was adapted by trainees to their specific teaching context; the experimentation was carried out in two 6^{th} grade classes, at about the end of the school year.

Pupils (21 in one class, 26 in the other one; aged 11-12) were asked to measure some quantities related to their bodies (height, weight, length of an arm or a foot,…), comparing the quantities thus obtained (by means of a calculator in one case and of EXCEL in the other) in a search for constants or significant correlations. In the light of Leonardo’s proposals, they were asked to answer the following questions: Is there a constant ratio between some anatomical measures? And between height and weight? If the ratio is not constant, what does that mean and what indications does it give us?

At the end experimenters reported on their activity to other trainees and commented upon some hypotheses that were formulated during the preliminary discussion.

Finally some activities for deepening the study were proposed: among the most significant, examples are the study of a link with Science teaching, through the study of children’ physical development or rather a link with History teaching, searching for still existing traces of measure units imposed on local markets.

PRESENTATION

The theme of Measure offers multiple hints for activities that present pupils with mathematical contents linked to real and concrete activities, not constrained to the restricted areas of lengths, weights or surfaces, even though in the actual teaching practice we will most probably be obliged to work with such quantities, due to practical issues. Everyday life continuously poses us measuring problems, but also portrays situations in which measure takes shapes and meanings that differ a lot: Stock Exchange indexes, dress and shoe sizes, money, statistical indexes… We are constantly offered new and more precise measuring instruments, in a search for the greatest possible exactness: a striking example is the move from manual timing to electronic timing in sports activities (athletics, skiing and so on). These initial remarks show some aspects of measuring and a possible didactical approach can be identified.

If measuring means identifying a number expressing the ratio between a given quantity and a prefixed unit of measure, for each object one can make different measurements, depending on the "quality" of the object one wants to measure: different types and with different "instruments", ranging from the human eye to the most sophisticated devices. Although highlighting the important and essential role played by instruments we want to "demythologize" them. No perfect instruments exist and all the measures we take are always approximate. Likewise there are both measurable and non measurable quantities, at the endpoints of the measuring scale etc.

It is also well known that before coming to the current standard units of measure, humanity went through a long phase in which units of measure were fixed arbitrarily and only for commercial reasons, there was a standardisation in single markets. Using different units creates no problems if our only aim is to make a comparison or an internal ranking; but if we want to communicate the result obtained to others or simply to compare objects located in different places, we experience some problems. Hence the need to use a standard unit, equal for everyone, to identify exactly the link between different measures and communicate to others the result of a measurement. By using conventional units of measure, we get to the correct definition of measure and to its denotation by means of a number followed by a unit of measure (cm, kg, l, etc.) to identify univocally a quantitative feature of the object (dimensions, weight, capacity-volume, etc.).

Nevertheless, for some pupils this path is still to be completed; more generally many students do not have an intuitive image of the value of a measure (*How wide is a window? How tall is a house? How many bottles of water would we need if we wanted to transform our classroom into a swimming pool?*).

The starting point of a specific didactical path might be a reflection on the appropriate use of some specific terms, clarifying the ambiguity of certain words taken from spoken language with respect to how they are used in the different domains (for instance in common language we use the terms "big, little" as referred to either dimensions or age, "capacity" with relation to intelligence or containing, etc.). A possible interdisciplinary activity will be a revisiting with the History teacher of the historical path[8] that starting from the Commercial Revolution in the XIII-XIV

[8] The region, as well as many other regions in Europe, is rich in examples of first "conventional" units of measure on local basis: length, weight and capacity standards are located in squares where local markets were held.

centuries led to the Weights and Measures Committee constituted during the French revolution[9] and later to fix the current conventional units of measure.

Picture 12. Ancient length and volume units from market squares

We interpreted the activity on anatomic measures that we piloted in this key: on one hand it offers ideas for a historical view and on the other hand for a reflection on the sense of measuring.

The activity was presented (two hours in the course of Mathematical Education) after students were given time to study the use of EXCEL worksheets and had dealt with measures of numeric synthesis, regression line and correlation coefficient in the Statistics course. Even though the latter topics were not central to our activity it is anyway good for teachers to know at least the basic notions, in order to gain a deeper understanding of the links between involved data.

ACTIVITY WITH TRAINEES

In the introductory lesson SSIS students were presented with some examples of "conventional" measures, locally adopted in past centuries, before the adoption of the International System of Measures. Then they were asked to sketch some possible activities that could help pupils reflect upon the usefulness of standard conventional measures, going along the path that led to the identification of some key lengths.

During discussion trainees remarked that often in the past men made use of parts of their own body to measure lengths or of the whole body as "reference weight". Of course this is due to the fact that it is "convenient" to carry one's measure instrument all the time (in the same way, the arm is better than the chest circumference as an "instrument" to measure lengths…); the subjective values obtained through these instruments, though, invalidated measure and suggested the opportunity of finding ways to make measures objective. But do "anatomic constants" exist? One would reasonably say no, and it would be correct if we look at the values of involved quantities. Trainees were nevertheless faced with the fact that the situation might

[9] Committee which involved also great mathematicians like Lagrange, who chaired it: it is to Lagrange that we owe the decisive boost to the adoption of the decimal system.

change if one looks at ratios between quantities[10]; we referred to the well known text by Leonardo da Vinci about the Vitruvian man.

It was noticed how most of Leonardo's statements concern not measures per se but rather ratios between them. This hint is actually very natural: any work on the concepts of quantity and measure, either related to mathematics (length of a segment, width of an angle), or to sciences (mass, weight, pressure, absolute and relative atmospheric humidity), soon leads to talk about ratios. Measure itself is conceptually a ratio. The same happens if we report data collected during a statistical study, with determination of mean, median and calculations of percentages.

However, work on ratios often involves heterogeneous quantities and it is not always easy or possible to deal with new quantities that ratios define and the proper units of measure at the school level we are referring to. It is for this reason that the hint to work on homogeneous quantities (thus obtaining pure numbers as ratio) coming from Leonardo's text was extremely stimulating for the initial phase of the work.

Another interesting idea comes from the possible geometrical interpretation of the constant value assumed by these ratios: they would be directly proportional quantities, and this would be easily checked on the Cartesian plane, either directly or through the application of electronic worksheets.

Picture 13. Trainees are measuring each other

Trainees contributed to the selection of some sentences that seemed suitable for experimental verification:

"The length of a man's outspread arms is equal to his height";

"From the elbow to the tip of the hand will be the fifth part of a man"[11];

"From the bottom of the chin to the top of the head is one eighth of his height".

[10] Students were suggested to carry out an autonomous research using both their competencies in anatomy (we remind here that SSIS students involved in the activity were graduates in scientific disciplines, some of them in Biology and thus have notions of comparative anatomy), and collaboration with History, Arts and Sports and exercise teachers.

[11] This statement led straight to an interesting discussion: the first measures gave results that were totally different from those expected (the ratio was closer to 4 than 5). Only a more precise reading of the text and the examination of the attached drawing allowed students to notice how a preliminary decision on the meaning of "hand tips" was necessary to be able to measure correctly.

The remaining part of the lesson was used as a laboratory in which trainees measured one another, and then reported measures on some EXCEL worksheets in order to verify whether Leonardo's hypothesis was correct.

"*Vetruvio architetto mette nella sua opera d'architettura che lle misure dell'omo sono dalla natura disstribuite in quessto modo. Cioè, che 4 diti fa un palmo e 4 palmi fa un pie: 6 palmi fa un cubito, 4 cubiti fa un homo, e 4 chubidi fa un passo e 24 palmi fa un homo; e cqueste misure son né sua edifizi. Se ttu apri tanto le gambe che ttu cali da capo 1/14 di tua alteza, e apri e alza tanto le braccia che colle lunghe dita tu tochi la linia della sommità del capo, sappi che 'l cientro a sinistra e a destra della scala metrica delle stremità delle aperte membra fia il bellico, e Ilo spazio che si truova infra Ile gambe fia triangolo equilatero diti palimi palmi diti. Tanto apre l'omo ne' le braccia, quanto è lla sua alteza. Dal nasscimiento de'capegli alfine disotto del mento è il decimo dell'alteza de l'uomo. Dal disotto del mento alla somità del capo è l'ottavo dell'alteza de l'omo. Dal disopra del petto alla somità del capo fia il sexto dell'omo. Dal disopra del petto al nasscimiento de capegli fia la settima parte di tutto l'omo. Dalle tette al di sopra del capo fia la quarta parte dell'omo. La magíore largheza delle spaffi contiene in sé* (*la oct*) *la quarta parte dell'omo. Dal gomito alla punta della mano fra la quarta parte dell'omo. Da esso gomito al termine della ispalla fa la ottava parte d'esso omo. Tutta la mano fa la decíma parte dell'omo. Il membro virile nasscie nel mezo dell'omo. Dal disotto del pie al disotto del ginochio fia la quarta parte dell'omo. Dal disotto del ginochio al nasscimento del membro fia la quarta parte dell'omo. Le parti che ssi truovano infra il mento e 'l naso e 'l nasscimento de' capegli e quel de' cigli, ciascuno spazioper sè è ssimìle all'orecchi*(*i*)*o, è 'l terzo del volto*[12]".

"Vitruvius, the architect, says in his work on architecture that the measurements of the human body are as follows that is that 4 fingers made 1 palm, and 4 palms make 1 foot, 6 palms make 1 cubit; 4 cubits make a man's height. And 4 cubits make one pace and 24 palms make a man. The length of a man's outspread arms is equal to his height. From the roots of his hair to the bottom of his chin is the tenth of a man's height; from the bottom of the chin to the top of the head is one eighth of his height; from the top of the breast to the roots of the hair will be the seventh part of the whole man. From the nipples to the top of the head will be the fourth part of man. The greatest width of the shoulders contains in itself the fourth part of man. From the elbow to the tip of the hand will be the fifth part of a man; and from the elbow to the angle of the armpit will be the eighth part of man. The whole hand will be the tenth part of the man. The distance from the bottom of the chin to the nose and from the roots of the hair to the eyebrows is, in each case the same, and like the ear, a third of the face[13]".

In the subsequent discussion students were invited to answer the following questions, mainly focusing on didactical aspects of the activity:

- What competencies are involved in this type of activity? Which pre-requisites are necessary? What kind of learning is promoted?

[12] Leonardo da Vinci, Le proporzioni del corpo umano secondo Vitruvio, disegno, 1485-1490 (Venezia, Gallerie dell'Accademia – Gabinetto dei Disegni e stampe); cat. 228

[13] From: *The Notebooks of Leonardo da Vinci, Vol. 1* (of a 2 vol. set in paperback) pp. 182-3, Dover, ISBN 0-486-22572-0 (J.P. Richter; original edition 1883).

- What difficulties have you met in this activity? Do you think pupils would meet further difficulties? How can they be helped overcome them?
- How much and which statistics are involved in the activity? How can we focus pupils' attention on the acceptable level of approximation?

The activity itself naturally led to a discussion about the range of precision for the measures found, due to the fact that Leonardo's hypothesis contains fractions, whereas matching them with the values found in the activity essentially depend upon the accepted approximation. This is an extremely delicate point, as confirmed by the repletion of the experience in the classroom: trainees had taken measures with a good approximation, whereas pupils' measures had a greater variability and thus required a further refinement before being used.

Once again trainees suggested how the research field could be widened: they proposed to calculate median, standard deviation and other synthesis measures of the measures they found that way.

The stimulating question "on the basis of these measures, what meaning do the expressions *tall, short, fat, thin,* … assume in this sample?" showed how other measures are not only possible, but indispensable in this context, to answer questions and how it seems natural to introduce ratios between non homogeneous quantities and therefore dimensional units of measure. For instance to define fatness or thinness of a individual one cannot do without introducing the concept of body mass, expressed in g/cm. Moreover, this need might be emphasised when the declared objective is to introduce theme of ratios between non homogeneous quantities.

Before getting into the experimentation of the activity in the classroom, we discussed whether it was appropriate from a psychological viewpoint to deal with body issues with teenagers; trainees designed some didactical expedients to involve everybody without embarrassing anyone. It should be also noticed that (as it often happens) work in class strongly highlighted some of these difficulties, underestimated by trainees. It was only where the teacher himself played the game, letting students measure him, that difficulties were completely overcome, with consequent positive effects on both the success of the activity and the general climate in the classroom.

PILOTING IN THE CLASSES

Among the SSIS students two volunteered to experiment the activity in their classes. The plan for the proposal was agreed during a collective discussion, and adapted to the different classes and to the ongoing teaching schedule. Trainees that followed the piloting (the class teacher and another trainee) were asked to pay attention to points highlighted in the discussion, also to verify hypotheses made about difficulties and about the meaningfulness of the activity.

Since the experimentation was implemented at the end of the school year, some activities were slightly reduced in favour of others which seemed more urgent or meaningful.

In what follows we report a summary of trainees' final reports.

Grade 6, 3 hours work, 26 pupils involved

The activity was carried out in the last semester of a grade 6 class, as an instrument to recall fractions and some statistical synthesis measures. Carrying out the activity at school, where pupils measure themselves and measure one another, is highly engaging and enacts a sort of "emotional mobilisation" that favours the teacher's job.

In taking measures some natural problems immediately arose, thus highlighting how some issues can be addressed only in a conventional and agreed way. For instance, how do you calculate the distance from the hand tip to the elbow: from the inside or from the outside? Similar difficulties were met to measure length of the feet (of course there were pupils who did not want to take off their shoes …) and also the body height of those who wanted to keep their shoes on. And pupils immediately found out that measures are … all different, also beyond those errors they could easily spot because they were linked to the use of inaccurate instruments (ruler, T square, etc.): it was not possible for the same boy to be 154, 156, 158, 159 cm tall at the same time! This fact (intrinsically more telling than a long dissertation about measurement errors and precision of instruments...) gave rise to a lively discussion, at the end of which it was agreed that three classmates would make each measurement and the "official" measure to be taken would be the median of the three measurements[14].

Pupils were particularly struck by the fact that in their class the ratio between foot length and height has a percentage frequency of 78% on the value 0.15 (notice that 1/7 is about 0.142857…). For some reason this discovery amazed them more than others; anyhow, following this route (and with similar precision…) they verified the validity of the different ratios proposed by Leonardo.

Grade 6, 3 hours work, 21 pupils involved

The class was high-achieving and the teacher thought they did not meet great difficulties with fractions and ratios. He wanted to introduce the theme of proportional quantities: in order to do so, he used some data provided by the other class colleague, inviting pupils to experiment with new ratios. In order to foster the need to introduce dimensional quantities (in this case g/cm, as agreed with SSIS trainees colleagues), the teacher proposed the study of the ratio between waist measure and body height.

This ratio presented a greater individual variability in qualitative terms, although it got to the value of 0,48 (defined as a "chubby belly"…) in 43% of cases.

The teacher then asked pupils how they could give a more precise idea of more or less "chubby belly". Pupils answered that "it is enough to look at the weight!"; the teacher did not object and invited them to carry on.

[14] An interesting question was dealt with by trainees when the experience was reported to them: was this the "real measure" of the object being measured? Pupils had assumed this value as correct but no one could exclude further errors …

In the school nursery room, equipped with scale and a height measuring instrument, each pupil was invited to take off their shoes and get on the scale to measure their own weight, helped by a classmate. Only few pupils (the tallest and smallest and the fattest and thinnest) were embarrassed; initially some asked the teacher to record their measures confidentially, but then they were carried by the general enthusiasm, helped by the fact that the teacher himself (obviously the tallest and chubbiest) volunteered to be measured.

A table with name, weight and height of each classmate was filled in. Initially weight was recorded in kg and height in metres; later, when they came to ratios, these measures were turned into grams and centimetres. In Table 1 there are some examples of the quantities found.

Initially the table did not include the ratio column. To introduce this concept, the teacher deliberately went back to fatness and thinness, asking pupils to agree on who were fats and thins in the class. The request raised heated arguments: Claudia and Chiara P. both weighed 49 kg, but it was clearly visible that the former was much thinner. Hence weight alone was not a good indicator of “fatness”; however it was easy to see why, in the case of the two girls: Chiara was 1,58m tall whereas Claudia was only 1,50m. But the comparison was not clear for all pairs and in any case it was a qualitative comparison, while the teacher insisted to get a quantitative comparison, through a measure of everyone’s “fatness”.

Interestingly, although pupils were working with fractions in that period and had made exercises about ratios, and although the class was rather high-achieving as we mentioned earlier, no one of the pupils thought about dividing the weight by the height, probably due to the fact that the quantities involved were not homogeneous. Finally the teacher proposed the division and, with the support of a calculator, the third numerical column of Table 6 was completed.

Pupil	BODY WEIGHT (in grams)	HEIGHT (in cm)	RATIO (approximated)
Alessandro	46.000	142	323
Chiara L.	34.500	147	234
Chiara P.	49.000	158	312
Claudia	49.000	150	326
Ester	26.000	122	213
Fabio	50.000	144	347
Francesco	42.000	145	289
Franco	31.000	141	219
Gianna	61.000	151	403
Giorgio	50.000	153	326
Giovanni	41.000	142	288
Giulia	45.000	148	304
Loretta	35.000	138	253
Marcello	45.000	150	326
Marco	41.000	142	359

Marta	33.000	136	242
Maurizio	59.000	148	398
Michele	48.000	145	331
Prof.	84.000	174	482 **MAX**
Sunita	51.000	153	333
Susanna	30.000	142	196 **MIN**
Yu Lin	38.000	144	263

Table 6. Weight and Heights of pupils in a piloting class

Pupils were then asked to discuss the meaning of Chiara P.'s 312 g/cm against Claudia's 326 g/cm. Some expressed the idea that, since dividing 39000 grams by 138 cm is like cutting the considered weight into 138 parts, each 1 cm tall, our ratios could be viewed as the expression of the weight of a "steak that could be taken from each of us" through horizontal cuts. The teacher accepted the idea, pointing out that it was necessary to think about pupils as having a perfect cylindrical shape, constituted by homogeneous material, or rather clarify that it was an "average" steak (and this allowed him to recall the concept of "arithmetic mean").

In the end each pupil compared their "average steak" with others'; of course the greatest steak was the Mathematics teacher's one: almost half a kilo!

At this point it was proposed to both classes to represent pairs of quantities studied up to then, as pairs of coordinates of points in a Cartesian plane. A first trial was made manually: of course the rather small range of measures required some expedients, but it was a good chance to present to pupils the use of graph paper sheets and the concept of representation to scale. Here teachers saw the results of previous discussions about approximation errors: also the most untidy pupils paid considerable attention to precision in the representation.

Representation of the relationships suggested by Leonardo did not create further difficulties and pupils could find out the existence of the predicted direct proportionality. On the contrary, more problems arose in the representation of the relationship weight-height, starting from the need to establish different scales for the two Cartesian axes. At the end of the work they got a multiplicity of points which highlighted the fact that there exists no clear proportionality between the two quantities[15].

A POSTERIORI ANALYSIS OF THE SESSION IN THE CLASSROOM

After trainees reported on their classroom experimentations, the discussion focused on the difficulties they met and on possibilities of further developments for the theme.

[15] Work stopped here, but it would not be difficult at this point to introduce the correlation coefficient and draw the possible regression line between the two quantities by means of an electronic worksheet. These topics go beyond the school level we refer to. It is worth reporting here a similar experience with 15 year old students, in which we tried to search for possible correlations between height and weight (about 0.8 for that class), between height and mean of grades obtained in Mathematics (far less than 0.5) and between weight and grades obtained in Mathematics (not trivial in this case: more than 0.65 …) with interesting hints for discussion about the meaning and value of statistical correlations

It was noticed how the activity naturally raised mathematical and statistical issues (approximations, graphical representations) that it is difficult to deal with in other ways.

PROPOSALS FOR FURTHER DEVELOPMENTS

At the end of the final discussion, one of the trainees (recently become a mother) proposed an activity that might represent a natural development of the experimented activity. She brought copies of two standard paediatric files, with indications of development percentiles (different for males and females) for the two quantities under consideration. On these, one might plot down obtained measures, inviting pupils to get their own development data at different ages at home, and report them on the graph.

This activity might be used in Mathematics to introduce the concepts of graph, function, percentile on a concrete example, extremely close to pupils' interests; in Science (possibly with the support of a doctor) the activity might be used to introduce the concepts of "body development" and sense of time and individual variations in this development.

One more proposal was made to extend the experimentation, both verifying other statements made by Leonardo about pupils' anatomy, and, through collaboration with the Art teacher, examining these ratios on classic statues (possibly drawing on Internet sources too). Following this path, it was suggested that both using Internet sources and organising simple tours around the Region, students might be led to discover traces of the first "locally conventional" units of measure, comparing for instance the "arm" locally used as unit of measure in different markets with the actual length of an arm, moving then to examine statistics about the increase in man's average height and weight over time. A search conducted on old public or family photographs might be a way to help the pupils link observations and verify hypotheses.

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Second piloting

by Yves Alvez*, Jean-François Chesné* and Marie-Hélène Le Yaouanq*

THE TRAINING INITIATIVE

This activity deals with the collection and analysis of data, and it demonstrates how the trainers want to link several modules together in the training of mathematics trainee teachers.

Depending on the year, between 50 and 80 maths trainee teachers for colleges and lycees (PLC2's) go through the IUFM training in Créteil. It includes a 42 hour-long module of mathematics classroom practices (module A). This module aims at accompanying the trainee teacher, in liaison with the educational advisor tutor, in their discovery of the teaching profession and at facilitating the build-up of their professional practice by providing them with teaching tools as well as pedagogical and didactical elements of reflection: (curriculum, working out modes of progression, preparation of sequences and sessions, evaluation, awareness of pupils' diversity, mathematical content, specific work in algebra or in geometry…).

It also includes a probability-statistics module (module S) made up of 12 compulsory hours and six optional ones. The aim of this module is to encourage trainees to give statistics their real place within mathematics teaching in upper secondary schools.

The organisation of this module requires independent computer work from the trainees in order to familiarize themselves with the use of the spreadsheet and graphics software from statistics (integrated functions, addressing, notion of variable, algorithmic aspects, etc.) and sets out examples of classroom practices in order to explore graphical tools and descriptive statistics methods: numerical and graphical characteristics, comparisons and interpretations.

As in any training initiative, the description and reflection involved will be carried out at two levels: that of trainers towards trainees, and that of trainers interested in trainees' practices and their effects upon pupils.

We will therefore specify our objectives and prior expectations regarding trainees, and then we will introduce the training initiative as it has been carried out this year i.e. its development from beginning to end. This will be followed with an analysis a posteriori, still on two levels, that of the session delivered by a trainee in the classroom and that more global of the initiative in its entirety. Finally, we will formulate some perspectives offered to us as IUFM trainers in Créteil and as members of the LOSSTT-IN-MATH project.

* Institut Universitaire de Formation des Maîtres – IUFM of Créteil, France.

A PRIORI ANALYSIS

This theme (proposed by the IUFM in Créteil in the LOSSTT-IN-MATH project and strictly connected to the proposal of Body Measures) combines two key aspects of statistics teaching in upper secondary schools. The first one can be defined in terms of mathematical content to be taught (see curriculum and official directives). The second one aims at developing trainees' critical reflection and their ability to distance themselves from the content. The integration of new technologies, to which half the module time is devoted, figures of course in this training initiative.

As for the procedures, they are worked out with work on practices in mind and not mere discourse, with the will to act upon both the cognitive and mediating components of teaching.

To be more specific, our objectives in this initiative are:

- To familiarise trainees with the use of a spreadsheet and to show them its value as a teaching tool.
- To get the trainees to act as pupils by asking them to do the work that they are then encouraged giving to their own classroom (modelling strategy).
- To introduce the notions of arithmetic mean, standard deviation and coefficient of variation of a set of data.
- To introduce an historical document ("Leonardo's man") and use it as an aid in the study of the mathematical notions aimed at.
- To get the trainees to deliver effectively this session with their pupils.
- To develop a pertinent use of the calculator in the classroom.

DEVELOPMENT

The training initiative takes place in 4 stages.

- Two sessions are devoted to mastering the use of the spreadsheet.
- During a subsequent session, the trainees are given several activities, among which is "Leonardo's man".
- A trainee delivers a session in the classroom.
- Return to all the trainees for feedback.

1st stage

Two 3-hour sessions are devoted exclusively to introducing trainees to spreadsheet functionalities (and other software specifically related to statistics). During the course of the first session, the trainers introduce the trainees to the technical aspects or components of a spreadsheet and to its key didactical characteristics as specified in the curriculum, and then the trainees are given a number of activities (involving in particular work on addressing). The second session is devoted more specifically to use spreadsheets in statistics (statistical functions and simulation of random experiments). Two trainers for a group of about fifteen trainees run each session.

2nd stage

During the course of a training session, trainees are given three activities focussing on the evaluation of the dispersion of a series and on the notion of randomness. They act as pupils in each one.

3rd stage

The filmed session takes place in a trainee volunteer's classroom, without any institutional evaluation. During a previous meeting, immediately before the session, the trainee has introduced her class and has presented her project to one of the trainers. In the same way, after the session, she will make some 'on the spot' comments.

4th stage

In order to adhere to the development of the training plan, the return stage to a training module took place rather late. The trainee who delivered the session shares verbally with the others her feelings about being filmed and gives a short analysis a posteriori of the session. The other trainees intervene to ask questions. As in the "Introduction to proportionality in geometry" initiative, no training work from the video is carried out with the trainees as this mode did not figure in the initial training plan and could not be inserted here.

"HOMOLOGY": TRAINING BY MODELLING

A quick reminder of the notion of modelling

The trainers transmit their own conceptions of mathematics teaching by putting them into practice in the sessions they deliver. The trainees are then expected to in turn implement in their own classrooms the sessions they have experienced as pupils. Modelling strategies differ from cultural strategies (where the trainer passes on a piece of information), from demonstration strategies (where the trainer transmits a teaching practice by implementing it effectively in his/her classroom) and from transfer strategies (where the trainer transmits referential knowledge about teaching and tries to harness the transfer phenomenon carried out by the trainees).

The training session (45 minutes) [*This stage was videorecorded*]

A trainer hands out the "Leonardo's man" drawing and its accompanying text to the trainees (see page 145). The trainees go through both documents quickly, and then the trainer suggests that they should focus on one of the statements in the text "Tanto apre l'omo nelle braccia quanto è la sua alteza – The length of a man's outspread arms is equal to his height". Each trainee is then asked to pair with another and measure each other's arm span (A) and height (H). The trainer shows the way to proceed. Each trainee then calculates the ratio $R = A/H$ to the nearest 0.01 and comes to the board to write out anonymously the values A, H and R thus obtained. The trainer points out experimental precautions. Meanwhile, the other trainer captures on a spreadsheet the data written down on the board. Afterwards the interest shifts to the statistical series of R values and it is then that the question of the dispersion of the series is brought up. Its spread having been determined, the trainees then suggest calculating its mean and standard deviation: the trainer takes this opportunity to point out the difference between the standard deviation of the sample and that of the population. The trainees perform all the calculations using a calculator, while one of the trainers carries on doing the same on the spreadsheet. In order to refine the evaluation of the dispersion of the series, one of the trainers suggests that the trainees should calculate the coefficient of variation $\sigma/\bar{x}$ (without dimensions) and asks them questions about the possible interpretations that can be made from these three parameters. Has the aim of this activity, i.e the search for a possible experimental validation of the Leonardo's statement, been reached? Which significance can we attribute to the value of the coefficient of variation obtained (≈4%)? Two more activities in this module (one involving the age of the trainees, the other tables of random digits) will enable us to produce some answers.

At the end of the activity, the trainers introduce the LOSSTT-IN-MATH project and ask for volunteers to deliver a session with the pupils. The trainers suggest to the trainees that they should adapt this session for their own classroom (content and procedures: for example the concept of standard deviation does not figure in the college syllabus), based on what they have experienced and felt.

THE SESSION IN THE CLASSROOM (*50 MINUTES*)

Presentation of the context [*This stage was videorecorded*]

The filmed session takes place in the framework of a 'Path of Discovery' in a third year class at Jean Charcot de Fresnes College, in the Val de Marne department. Charcot College, which counts 330 pupils for 25 teachers, is a "smallish" establishment.

With full teaching time allocated, these 'Paths of Discovery' are compulsory. They make up for 2 hours of the weekly timetable of all pupils in the middle cycle (second and third years of lower secondary college). They are added to the compulsory subjects, combining at least two disciplines structured around one common theme belonging to one of the following 4 fields:

- Natural and human sciences
- Arts and humanities
- Language and civilisations
- Design and technology

Each 'Path of Discovery' lasts between 12 and 13 weeks, periods for presentation, learning, work and evaluation included. So, over a school year, the pupils in the middle cycle attend two 'Paths of Discovery' (POD).

The POD on which the trainee is working, in collaboration with a French Teacher, covers the following topic "Journey around the World". The second part of the study uses statistical data relating to the European Union with the following objectives in terms of mathematical content:

- To read and interpret a graph or a diagram.
- To calculate integers, frequencies, cumulated frequencies, cumulated integers and means.
- To represent a statistical series as a spreadsheet or diagram.

The session takes place after studying the chosen topic. Therefore, the pupils can a priori make use of the above-mentioned tools. Finally it must be noted that the pupils attending the session do not all come from the same class: they come from different third year classes and are regrouped for 2 hours a week (one hour with the French teacher and one hour with the maths teacher).

Development of the session

1st period (15 min)

The teacher hands out the first worksheet to her pupils and projects Leonardo 's drawing. She asks a few questions about Leonardo, and then about the drawing itself. The pupils have to go over the square, the arms and the man (from top to toe) in colour. The teacher too goes over the lines of the projected drawing and then asks the pupils to reflect upon what they are being shown. With sustained help from their teacher, the pupils suggest that the man's arm span is equal to his height. One pupil is sent to the board to write the concluding sentence: "A man's height and arm span are equal".

2nd Period (15 min)

After the reading of the instructions by a pupil, the teacher herself, together with another pupil, performs the tasks expected from all of them. The pupils then get up to pair and measure one another. The teacher allows 4 girls to remain together, she circulates between the pupils, providing help to some. As soon as the measurements are taken, the pupils go back to their seats to calculate the ratio.

3rd Period (20 min)

The teacher hands out the second worksheet to her pupils and writes out on the board the ratios obtained. It is then strange to notice that several pupils give 1 as the A/H value. The pupils determine the minimum and the maximum of the obtained series, and then calculate its mean (which is actually 1!). The teacher then tries to get her pupils to go over what they have just done, by questioning them in particular about the significance of A/H. The period ends with the following writing on the board: "The ratios are approximately equal to 1. Arm span and height are therefore close".

The session ends with the teacher giving homework to be completed for the following session.

A POSTERIORI ANALYSIS OF THE SESSION IN THE CLASSROOM

The pupils are placed in a U shape in the room.

Pupils' worksheets have been well prepared, the scenario (see page 145) has accurately foreseen the different periods planned by the teacher, and the timing of the session is defined.

The collective instructions given by the teacher are very strict, her tone of voice very firm. Individual help is however very frequently and kindly given.

The ratios having been written on the board, pupils are mostly using their calculators correctly to determine the mean of the series of ratios. There can be doubts however as to whether the choice of A/H ratio was a priori a pertinent one for third year pupils: several questions from pupils throughout the session reflect a rather vague representation of the relation A/H = 1 with A = H. The choice of precision to be given to the A/H ratio added to the absence of a unit does not seem to have been apprehended by the pupils.

The actual development of the session adheres to the teacher's initial plan. (During the meeting that followed, immediately after the session, she declared herself "pleased with the way the session went").

The transition from observation of the drawing to conjecture is largely made with a great deal of help from the teacher, in a pretty short space of time, and it can be said that in the end this stage was not student based. The teacher starts systematically with the answers which she wishes to hear from her students "the *Topaze* effect" is here manifest. You may think that this is her way to manage the development of the session. But in the end, what should have appeared to the pupils as a hypothesis to be verified experimentally turned in fact into a certainty to adhere to at all costs, even if it means retaking or readjusting the measurements, to the nearest mm in the case of some pupils.

A POSTERIORI ANALYSIS OF THE TRAINING INITIATIVE

A modelling strategy aims mainly at presenting a session that the trainers consider suitable for delivery in the classroom. In any case the content and development chosen were regarded as "labelled"/ received with a seal of approval by the trainees, without the trainers' choices being necessarily made explicit. Moreover, in giving the trainees the role of pupils, the aim is to facilitate their questioning of the prescribed tasks, which would possibly not have taken place otherwise.

Now, what transpires from the session delivered in the classroom? Training documents were properly re-used, the initial management of pupils was put back to good use but the didactical issue at stake, introduced in training, was totally missing: the trainees were to wonder about the validity of Leonardo's statement using statistical tools available to them, the pupils tried merely to "enter Leonardo's square". But could the session, as constructed by the trainee volunteer, allow them to do otherwise? Indeed, it would seem that it was difficult for her to adapt the session presented in training, aimed at 6th year pupils of the upper secondary school and beyond.

She perceived the absence of standard deviation as a tool (and as a result the coefficient of variation too) as mere removal of the superfluous, when in fact it cast serious doubt over the choices made for such a session. It is reasonable therefore to think that experimental work on measurements and the calculation of ratios should be carried out before the introduction of Leonardo's drawing and text, so that the pupils can really be made to conjuncture the existence of a law, which anyway with their current level of knowledge, they can neither validate nor invalidate. This choice together with a coherent talk would by the way enhance the teachers' critical approach and ability to step back, aimed at during training.

COMMENTS

The main difficulty facing trainers during maths teachers' initial training is to know what in the practice of a novice teacher falls within professional learning generally, and more specifically what can be inferred by out of school training. It is well known nowadays how important teachers' metacognitive representations (on mathematical content, on teaching this content, on the role of mathematics in school years, on teacher/pupils relations …) are in their practices. It is also known that not all 'ingénieries' (meaning: precisely constructed teaching projects), are applicable in practice and among those that are, not all are put into practice by the same teacher.

Therefore, a modelling training strategy could provide a compromise, offering both the possibility of a classroom situation, and the opportunity for trainees to question positions they may have adopted more or less consciously, regarding mathematics and their teaching.

The training initiative presented here seems to show that if such a strategy allows the modification of practices in pupil management and in the choice of activities, if it enables trainees "to do something which really works in the classroom" (according to them), it does not seem however sufficient: the trainee has failed to carry out properly the necessary work for successful adaptation. Could it have been done at this stage? Can training facilitate it? Before or after experimentation in the classroom? How can the video be exploited in initial training without putting off the trainee being filmed? And on a wider scale, what are the constraints of the observer/trainer's initiative? We hope that the confrontation and mutually beneficial collation of presentations from the different partners in this project will enable us to bring some answers to all these questions.

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Third piloting (at Skårup Seminarium) and conclusion

by Brunetto Piochi

One of the main difficulties facing trainers during mathematics initial training is to put together the general knowledge of the matter (which the trainees must have obtained in earlier studies) with the way of "teaching how to teach" such knowledge to pupils. It is well known nowadays how important teachers' metacognitive representations (on mathematical content, on teaching this content, on the role of mathematics in school years, on teacher/pupils relations …) are in their practice. Therefore, a modelling training strategy could provide a compromise, offering both the possibility of a classroom situation, and the opportunity for trainees to revise specific contents and to question positions they may have adopted more or less consciously, regarding mathematics (or specific aspects of it) and their teaching.

Both partners piloted the experience in such a way that it became a lesson "by modelling", i.e. proposing trainees to carry out the activity as it would be later proposed in the classroom. The discussion that followed made the trainees able to structure a proposal that took into account not only the main mathematical aspects of measuring, but also some (both practical and epistemological) obstacles identified by trainees themselves. The initial practical phase, measuring parts of their own bodies, worked as a motivating factor for trainees (as well as for students in the subsequent implementation in the classroom), but most of all it allowed them to experience most of the difficulties that were later met in the activity with pupils: for example trainees themselves felt some possible reticence when asked to communicate personal features. The whole activity enabled them to make a finer *a priori* analysis and, once they were in the classroom situation, they could react more promptly and appropriately to unforeseen obstacles.

The differences between the two partners' piloting mainly lay under the following aspects:

- *the phase of data collecting*: IUFM exploited this activity to provide both an example of how to collect and analyse statistical data; SSIS left trainees more organisational freedom, since the aim was to structure an example of laboratory-like activity (for both partners this was also a good chance to stress that pupils can be allowed to "move and act" in the classroom)

- *opening and linking this activity to other themes*:
  - for IUFM: the use of software, reading of an historical document and education to citizenship;
  - for SSIS: the general approach to the history of measure and the introduction of ratios between non-homogeneous quantities.

The choice made by SSIS not to structure rigidly the analysis of collected data allowed for a less complicated and more involving work with trainees in the initial phase, but turned out to be less efficacious in the phase of comparison of classroom experiences. However this could not be avoided, due to the situation of the SSIS courses, characterised by the fact that many students (but not all of them) are already teaching in a class and the activity must be included in the actual curriculum. On the contrary it is the first teaching year for all the trainees in Créteil, who are part-time teachers under the supervision of a tutor: despite of such a guided training session, it was not easy for them to adapt the activity to their own classes.

Besides the two partners, the proposal was also partly piloted in Skårup Seminar (by Seminarielektor Helge Thygesen). In that piloting, the trainees worked only the first phase of the proposal: measuring themselves and discussing about discoveries. Firstly they aimed to verify the correctness of the hypothesis that "The opening of arms [and shoulders] of a man equals his height"; this was easily shown to be true by some measures and a simple EXCEL worksheet. But the trainees had previously worked with the golden section, so the teacher invited them to verify also that in general the ratio between one's height and the distance from one's navel and the ground equals the golden number. Indeed the evaluation following this activity led to the conclusion that it was not likely that there was such a standard ratio: the individual results differed too much from the expected results. But again interesting considerations came out on art and mathematics.

Lectures in Skårup ended by talking about measuring children in the school. The conclusion was that it was a very good idea to let children measure themselves. Trainees supposed that such an activity would definitely be of interest for the children and might generate an interest in old Danish measures like „favn" and „fod" that obviously derived from body measures.

However a point deserves to be stressed: the teacher must be careful when dealing with body issues with teenagers; in both full piloting activities, trainees were led to design some didactical expedients trying to not embarrass anyone, and piloting in the classroom revealed how hard psychological difficulties can raise. Difficulties were completely overcome only where the teacher deeply involved himself in the game; this was in fact the key passage: the teacher became an educational model, when faced to the acceptation of one's "difference" from standard ideas of body fitness.

The proposed activity substantially reached its aims, both for trainees and for secondary school pupils; the same happened in the piloting in Skårup. Surely such an activity (more in general any activity inspired to modelling strategy) enables trainees "to do something which really works in the classroom", according to the trainees in IUFM of Créteil. However it did not seem sufficient: the same trainees have often failed to carry out properly the necessary work for successful adaptation. Hence some natural questions arise, which concern mainly trainers, on how can teaching by modelling be enriched to overcome such a gap.

GEOMETRICAL PUZZLES

by Franco Favilli[*] and Carlo Romanelli[**]

INTRODUCTION

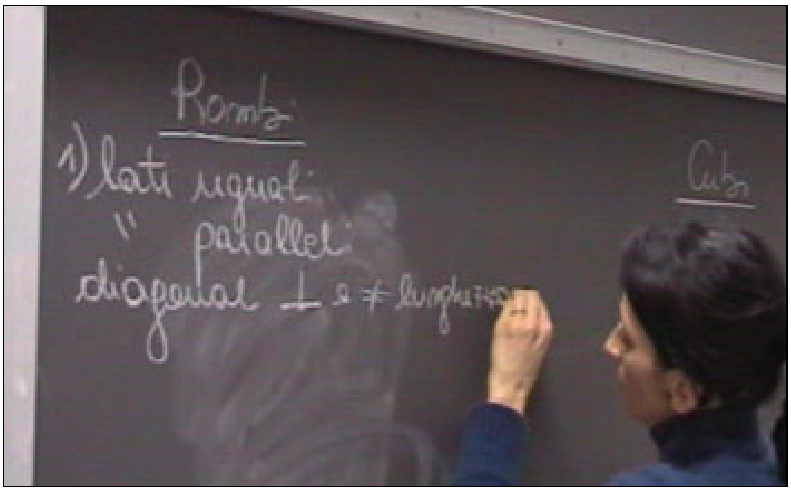
Geometrical discourse requires good knowledge and mastering of the terminology and the notions. On the other hand, the acquisition of geometrical concepts by the learners is facilitated when the communication is supported by the balanced use of graphical language, natural language and geometrical language.

In the proposed activity the learners are asked to work in pairs, one of them providing the other with a sequence of instructions for the drawing of a geometrical figure. Both learners are then asked to describe the figure and to define it.

This geometry teaching activity can represent an interesting opportunity to underline the need to promote the use of different registers of representation and their coordination through specific tasks aimed at converting one into the other.

Through this didactical proposal, the trainees can also directly realize how sophisticated and challenging the transition from the description of a geometrical figure to its definition is for the learners.

This proposal was prepared and piloted at the University of Pisa. It was also piloted, at the same time, at the University of Siena and, later on, at the IUFM of Paris.



[*] University Centre for Educational Research and Training – CAFRE, University of Pisa, Italy.

[**] Istituto Comprensivo "E. Pea", Seravezza, (LU), Italy.

Main piloting

by Franco Favilli and Carlo Romanelli

THE PROPOSAL

The *Geometrical puzzles* proposal seems to correspond with a good way of dealing with mathematical notions, introducing them through a nice mixture of theoretical and practical activities. Expanding and deepening the idea could easily lead the discussion well beyond the standard content of a lower secondary school mathematics curriculum. Its piloting requires therefore, first of all, the definition of specific didactic aims and the selection of only a few of the possible mathematical notions for introduction, or for further use (if they are already available to pupils).

At the beginning of the lesson, learners receive a sheet of paper (the Learner's Guide, see Appendix B on page 179) with some explanation about the content and the rules for the didactic activity. The basic rules are:

- Learners work in twos.
- A member of each pair is given a piece of paper with the name of a (plane or solid) geometrical figure that must be kept secret from the partner until the end of the activity.
- The first learner provides the other with a sequence of instructions how to draw the figure.
- Only *unitary* instructions that correspond with a single graphical activity of the partner are allowed. For example, the instruction "Draw a segment" is allowed, the instruction "Draw the axis of the segment AB" is not allowed, because it requires the determination of the middle point M of the segment AB first, and the perpendicular in M to AB after.
- Each instruction given/received is written on a sheet of paper by both learners.
- If necessary, an instruction can be repeated, but neither modified nor explained.
- For the drawing pupils make use of a sheet of squared paper and a pen (no pencil, ruler, compasses, etc.). No deletions are allowed.
- The in-progress drawing cannot be shown.
- When the sequence of instructions has come to an end, the final drawing is shown and compared with the name of the given geometrical figure.
- Both learners are asked to give the name, the description and finally the definition of the geometrical figure.
- Discussion in the whole class, rooted in the final drawings and the given directions, concludes the activity.

The same scheme should be used both by the trainers with the trainees and by the trainees with the pupils at school.

In Pisa, the plan of the piloting was designed and developed in the following scheme; the hours represent the duration of each step:

<table>
<tr><th colspan="10">Steps</th></tr>
<tr><td>Trainers (10h)

Preparation of the didactic proposal</td><td>Trainers and trainees (4h)

Introduction
Group work
Discussion</td><td>Trainees (2h)</td><td colspan="2">Trainees and pupils (2h)

Introduction
Group work
Discussion</td><td>Trainees (2h)</td><td colspan="2">Trainees and trainers (4h)

Reporting
Discussion</td><td>Trainees (4h)</td><td>Final report
Trainers (5h)</td></tr>
<tr><th colspan="2">Objectives</th><th colspan="7">In-context methodology</th><td></td></tr>
<tr><td></td><td></td><td rowspan="3">Thinking it over
Lesson planning</td><td>Trainees</td><td>Pupils</td><td rowspan="3">Thinking the lesson over
Report</td><td>Trainers</td><td>Trainees</td><td rowspan="3">Thinking it over – Remarks
Finalizing the lesson plan
Report to trainers</td><td></td></tr>
<tr><td>Short term</td><td>Knowledge
Competences</td><td>Knowledge</td><td>Knowledge
Competences</td><td>Socialization</td><td></td><td></td></tr>
<tr><td>Long term</td><td>Methodology
Socialization</td><td>Methodology</td><td>Socialization</td><td>Methodology</td><td>Methodology</td><td></td></tr>
</table>

Table 7. Piloting plan

General information

Number of trainers: 2

Number of trainees: 42

Number of classes involved in the piloting: 2 (2nd and 3rd class of lower secondary school)

Number and age of pupils: 24 pupils aged 12 (2nd class) and 22 pupils aged 13 (3rd class)

Number of adults in each classroom during the lessons: 2 trainees (present for the first time in those classrooms) and the teacher

Aims

The educational aims of the proposal can be roughly divided into general and mathematical aims.

Among the *general aims* we can consider:

- The development of awareness and critical attitudes towards the use of language and its interpretation.
- The awareness of how important it is to use specific and unambiguous language.
- The increase of the learners’ capacity to understand and to elaborate oral instructions.
- The stimulation of “critical” listening to instructions.

- The improvement of the ability of reading, understanding, respecting and applying the rules of the didactical activity.
- The acquisition of the notion of simple instructions.
- The ability to respect the pace of the classmates.
- The capacity to state the reason for the choices made and used during the activity.

Among the *mathematical aims* we can consider:

- The improved usage of mathematical language.
- The reinforcement of the knowledge of geometrical language.
- The improvement of drawing skills.
- The consolidation of geometrical knowledge.
- The ability to visualize three-dimensional objects from their two-dimensional representations and to represent solid figures on the plane.
- The capacity of describing basic plane and solid geometrical figures by pointing out the properties necessary and sufficient to define them.
- The development of the ability to find a proper balance between the description and the definition of a plane or solid geometrical figure.
- The awareness of the relevance of the definition in geometry.
- The capacity to compare and evaluate different kinds of input from the debate in the context of the correct construction of the concept of geometrical figures.

Assignments for the trainees

- Read the *Teacher's guide* (see Appendix C on page 181) very carefully!
- Make comments and proposals for the modification of the *learner's guide* you have received at the beginning of the activity.
- Are the rules stated there clear enough for the pupils?
- When developing the activity in your class, will you use a *log-book* (i.e. an account of how the lesson developed)?
- How much *time* should be devoted to the introduction, the development of the activity, and to the final debate?
- Should the didactical activity be presented to the pupils as a *role-playing game*?
- Considering that the *communication*, both active and passive, between pupils is quite relevant in this activity, what kind of *linguistic register* will you use with the pupils?
- Is it important that the pairs of pupils are matched evenly according to their ability?
- As regards the *geometrical figure* to be drawn, is it better to choose a figure that the pupils already know or a new one?

- What are the *advantages* and *disadvantages* of using a known or unknown figure?
- Is it better to use a *sheet of squared paper* or a *blank sheet*?
- Only *single* instructions are allowed. The notion of single or *unitary instruction* could be quite controversial: make your choice and explain it to the pupils. Why and how?
- What *prior knowledge* is required for the activity?
- List different possible *sequences of instructions* for the drawing of the chosen geometrical figure.
- Give examples of possible *ambiguous instructions* and consequent different drawings and *misunderstandings*.
- Are you aiming to introduce the *definition* of the given geometrical figure?
- How could you aid the pupils in the passage from the *description* of a geometrical figure, through (some of) its properties, to its *definition* by means of this lesson plan?
- What do you expect from the *final debate*? What role do you assign to it?
- Will you ask the pupils for a *final report* of the activity? From individuals or from pairs?
- Make comments and proposals for the modification of the *teacher's guide* you have received at the beginning of the activity.
- Did you fulfil the aims you had set for this lesson plan?

Assignments for the pupils

- Read the *learner's guide* very carefully!
- Be sure to have agreed with your teacher and your friend in the pair about the meaning of a *unitary instruction*.
- *It is not allowed to correct* a given instruction or a part of the drawing. Be very careful before speaking or drawing!
- When did you (the person who was receiving the instructions) *realize* what the geometrical figure was, that had to be drawn? Did it help you? Did you *stop executing* the actual instructions given by your friend (i.e. you started ignoring them)?
- How much was your *prior geometrical knowledge* useful?
- How difficult was it *to understand* the meaning of an instruction? Give at least one example.
- Did you receive any *ambiguous instruction*? If so, give an example.
- Did you (the person who was giving the instructions) *draw* the geometrical figure *before* you started giving the instructions or did you draw it step by step, thus doing what you were asking your friend to do? If it happened, did it help you?

- How difficult was it to find the *appropriate words* to give an instruction? Give at least one example.
- Are you *satisfied* with the experience? Why?
- Would you have liked it better to *exchange roles* with your partner?
- Was it difficult to realize that some properties of the given geometrical figure *depend* on some other ones? Give an example.
- Why, in your opinion, did your teacher *propose* this activity?
- Will you make a *report* of this activity?
- Make comments and proposals for the modification of the *learner's guide* you have received at the beginning of the activity.

THE PILOTING

The training session

At the beginning of the session the teacher trainers gave the trainees a blank sheet of paper and the learner's guide with the explanation of the activity that was about to start. The trainees were grouped in twos and they decided which one had to give the instructions and who had to receive them and make the drawing. The trainees who gave the instructions received a piece of paper with the word *rhombus* or *cube*.

As soon as the trainees started the activity, some of them asked for more explanation about the meaning of the term *unitary instruction*. The teacher trainers gave a few more, non mathematical, examples. Therefore the activity started half an hour after the beginning of the training session.

After another half an hour, when all the pairs had finished the instructing-drawing-describing-defining activity, the discussions within the pairs started. (see Photo 13)

Photo 13. Trainees discussing

The following debate in the class was organised in three stages:

- general issues, mainly related to the rules set in the learner's guide;
- comparison of the outcomes (list of instructions, drawings, descriptions and definitions);

- remarks and comments upon them, as regards the rhombus, first, and the cube, after.

The discussions and the debate lasted three hours, until the end of the training session. (see Photo 14)

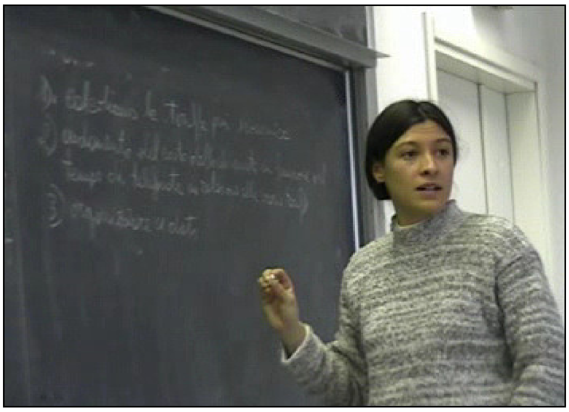


Photo 14. A trainee reporting for debate

Most of the trainees stated that they had realized how difficult it was to express in a clear and concise way even easy mathematical concepts and properties, as an instruction requires, mainly making only use of natural language and not of mathematical language. The most difficult task was to find a good balance between the two languages, also taking into account the restrictions given in the learner's guide with regard to the use of mathematical terminology.

As expected, most of the drawings were correct, even if many trainees stated that once they had realized which figure was to be drawn, they completed the drawing by (almost) ignoring the subsequent instructions from their partner in the pair. This could be considered a weak point in this part of the piloting of the proposal, because in such cases it was impossible to compare the specific instruction with the related drawing. It would therefore be important to emphasize more and focus the attention of the learners on the fact that any drawing has to correspond exactly to the given instructions, no matter if it is right or wrong.

Several ambiguous instructions were presented during the discussion, thus allowing the teacher trainers to recall to trainees' memory some mathematical concepts and/or to make a few of them better understood. It is important to outline here that the university career of all the trainees in the class was mostly in scientific subjects, but not in mathematics. They had studied only one or two courses of mathematics in their undergraduate studies.

The need for a better acquisition of some basic mathematical notions became evident, as expected, when the discussion started about the definitions of *rhombus and cube*. Quite often the relation between the description of the properties and the definition of a geometric figure was not clear at all. It was therefore necessary to spend much time on that and on the difference between the image/drawing of a geometrical figure and the geometrical figure itself.

The session in the classroom

Two trainees offered to pilot the lesson plan in a second class of a middle school, using the rhombus, and two others offered to pilot it in a third class, using the cube. Before the piloting trainees were asked again by the trainers to make their comments and remarks on the activity and the rules to give to the pupils (12–13 years old) for an effective and useful impact of the piloting, according to the aims originally given by the trainers and the possible modifications just agreed among the trainees. The learner's guide was slightly modified before entering the classroom.

The trainees decided to work with squared paper and no geometric tools, such as the ruler and set square (see Photo 15).

Photo 15. A pair at work respecting the given rules

The most relevant outcomes, similar to the ones with the trainees:

- It was necessary to explain better the meaning of unitary instruction.
- Some instructors said it was difficult to find the right words to structure some instructions (Photo 16), even if they had a clear idea of what they wanted their schoolmates to draw.

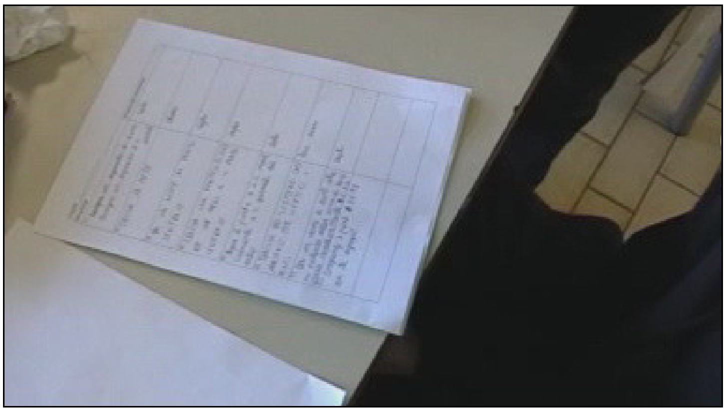


Photo 16. A list of instructions

- After a few instructions, most pupils successfully completed the drawing (almost) ignoring the remaining instructions: some drawings were correct even if the instructions were wrong (the important thing was to succeed, to win the game!).
- The use of the squared paper made the drawing activities easier.

- Several instructions were ambiguous and, therefore, misleading (see Photo 17).

Photo 17. Consequences of ambiguous instructions…

The most relevant outcomes, different from the ones with the trainees:

- From the beginning of the session pupils complained about the difficulty of the linguistic register used by the trainees both in the guide and the oral communication.
- Most pupils made good use of letters to identify the ends of the line segments.
- For most pupils to describe or to define a geometrical figure were equivalent notions.
- Low achievers greatly benefited from this kind of activity.
- The final debate in the classroom started with each pair presenting their activities to the whole class for the discussion.
- The change of roles within the pairs was an additional activity that pupils wanted to experience with different figures.

Feedback for the trainees

Two pupils participated in the feedback session as well as the two trainers and all the trainees.

The four trainees presented the piloting (see Photo 18) with the pupils to their colleagues, making comments and remarks, and showing video-clips from the class.

Photo 18. Trainees presenting the piloting to the colleagues

Most of the above outcomes from the piloting were introduced for specific discussion. However, it must be said that, while the pupils were active (see Photo 19) in the discussion, the non-piloting trainees just occasionally entered the debate.

Photo 19. Pupils actively debating with the trainees

Additional issues from the discussion:

- To find the appropriate way to introduce and motivate the aims of the activity.
- To identify the appropriate didactic contract between teachers and pupils for the implementation of the activity.
- To manage the time in the class (the piloting required more time than originally planned).
- To consider the prior knowledge of geometric software as a possible source of help, especially for the pupils who are asked to give the instructions.
- To mark the difference between the rigour of mathematical language and the "flexibility" of natural language.
- To decide how the transition from the description of a geometric figure to its definition can be demonstrated in a lower secondary school.

Second piloting

by Lucia Doretti[*]

PROPOSAL IMPLEMENTATION

The activity aimed at making trainee teachers reflect on the different languages, graphical and verbal, that intervene in geometrical discourse and determine its development through their coordinated interaction. It turned out to be an interesting opportunity to underline the need to promote teaching the use of different registers of representation and their coordination in geometry through specific activities for converting one into the other.

[*] Department of Mathematics and Computer Science, University of Siena, Italy.

Activity developed in the training session

Number of trainees: 18

Total duration: four hours (one hour for work in pairs – three hours for discussion)

The activity was set up following the directions provided by the piloting group in Pisa.

Trainees were grouped in pairs and within each pair the roles of receiver and provider of instructions were assigned. Each trainee was then given guidelines with instructions to follow together with the rest of the material. Before starting the activity it was necessary to clarify the meaning of the expression unitary (simple) instruction included in the sheet through examples to enable participants to draw the figure in successive steps. On the sheets given to those who provided instructions there were the words *rhombus*, *isosceles trapezium* or *cube*. At the end of the first phase, after one and a half hours since the working session began:

- each member of the pair had filled in their worksheet with either given or received instructions, indicated in writing possible comments and given a definition of the figure
- the sheet with both the name of the figure and the drawing made were shown.

The subsequent phase consisted of a collective discussion of the each pairs' work. The first comments by trainees were about the unexpected difficulties they met in this work, despite their familiarity with the geometrical figures considered. Everybody agreed that the role of *instructions provider* was more difficult than the role of *receiver and executor of instructions*: those who gave instructions had to refer to an image and name of the figure they could recall to their minds, to interpret that image both perceptively and cognitively, and formulate appropriate messages for its reproduction. Some declared they did not manage to complete the list of instructions (even after many attempts), others confessed they had difficulties about the way they had formulated them and were not sure they could be understood by their colleague. Then some lists of instructions provided by trainees (and reported on a transparency) were examined and commented upon together with the way these instructions were interpreted and graphically translated by the ones who received them.

In Appendix D on page 184, we report some of the worksheets commented in the session with trainees.

This was a chance to reflect with trainees on the two following aspects:

- mathematical language and its role in the process of knowledge construction;
- the role of definitions in geometry.

A. *Mathematical language and its role in the process of knowledge construction*

An appropriate use of language presupposes full awareness of the mathematical terms introduced and the indication of non ambiguous instructions that can be interpreted in the same way by anyone. In different cases instructions were given that permitted the drawing of figures with geometrical characteristics different from those we wanted to obtain (for instance the following instructions: *1. draw a segment; 2. draw another segment, perpendicular to it in its midpoint*, do not lead to a univocal identification of a rhombus, as they were meant to).

We had the feeling that the theme of the use of language could offer material for study, although not all trainees fully acknowledged the problem.

Somebody was puzzled by the critical analysis carried out on the given instructions, since they believed that "*although instructions are not very precise, the receiver is led to interpret them and complete them correctly*". In certain drawings we noticed that the figures were the intended ones, but it was not possible to deduce them from the instructions given.

The tendency in the presence of wrong or defective messages was to fill the information gaps on the basis of the global message, using perception and regularisation phenomena; in other cases, the initial visualisation of the figure, generated by the initially received information led the receiver to carry on with a correct drawing although instructions were not appropriate[16].

Discussion also led us to reflect on the difference between natural language, characterised by richness and variety of expressions, and mathematical language, in which each term has a specific meaning and this determines its use.

Finally it was remarked that in many cases, a 'stranger' element to the figure was introduced in giving instructions: we mean that we used information describing the figure as located in a standard way in the natural reference frame constituted by the borders of the page. This highlighted the existence of stereotypes in the mental representations of concepts and geometrical relationships (in fact, often instructions contain terms like "horizontal" and "vertical": "*draw a horizontal segment*", "*draw a vertical segment going through the midpoint*").

The experience made trainees reflect on the need to develop specific activities related to the learning and use of mathematical language in teaching. Although the correct use of language requires a considerable time to mature, it is nevertheless a fundamental instrument for knowledge construction.

B. *The role of definitions*

Another point for reflection was suggested by the request on the worksheet of giving a definition of the drawn or described figure. This request was interpreted in different ways by trainees: some only indicated the name of the figure and others listed many properties, often more than those needed to characterise it.

As an example, here are some definitions of rhombus given by trainees showing this aspect and which were analysed in the working session:

[16] In one comment we read *"Although indications were sometimes not very precise, I followed the indications in the most logical way* (*or the most trivial perhaps!*) *probably due to our geometrical notions"*.

- Rhombus: *"Plane geometrical figure, particular case of a parallelogram with opposite sides pairwise parallel and interior angles pairwise equal, and perpendicular diagonals with different lengths"*.
- Rhombus: *"Quadrilateral having all equal sides and equal opposite angles"*.
- Rhombus: *"Parallelogram with perpendicular and not equal diagonals"*.
- Rhombus: *"Quadrilateral with four equal sides, pairwise parallel"*.

We reflected upon the meaning of definition in mathematics and on the difference between properties that *describe* a figure and properties that *define* it. In particular, we discussed about how to "minimise" the stated properties in the case of the rhombus, so as to identify those necessary and sufficient to characterise it each time. We noticed that identifying necessary and sufficient properties is a fundamental moment in the construction of definitions, but it certainly marks a delicate and difficult transition moment for pupils. Hence we need to tackle the learning of properties with pupils to construct definitions that make use of specially designed materials[17] and/or of the Cabri software (the construction mode of Cabri highlights which necessary and sufficient properties were considered and therefore which definition is underlying).

We pointed out how the issue of definition is strictly intertwined with that of classification: i.e. properties expressed by the definition must allow one to include in a certain class only the objects that have those properties. We also noticed how definitions given by trainees often led to "classify by partition" (requiring that *a rhombus be a parallelogram with perpendicular and not equal diagonals* implies removing squares from the set of rhombuses). This was the chance to highlight the fact that Euclid in his "Elements" defines by partition: for example, the definitions of quadrilaterals he gives aim to determine a partition in the set of these figures. The choice we make nowadays privileges definitions that give rise to inclusive relationships, enabling a comparison between geometrical figures that highlights their analogies and differences. When it is necessary to distinguish among concepts (e.g., concave or convex quadrilaterals), one needs to draw on definitions by partition. We discussed the fact that classification by inclusion, although more complex, facilitates a deductive systematisation of notions (particular concepts are a subset of more general ones), is more economic than a classification by partition and makes it possible to give each geometrical object more alternative definitions (a *square* is *a rhombus with equal diagonals* or *a rectangle with all equal sides*).

In teaching, it is fundamental not to anticipate definitions before the construction of the environment in which they get a meaning.

[17] For instance "dynamic models": manipulating, analysing and describing them one can collect "elements" to construct definitions of figures referring to different properties.

Activity implemented in the classes

Number of trainees: 2

Overall duration of the activity in each class: two hours (one hour for presentation and implementation of the activity– one hour for discussion)

Activities were proposed by two trainees in their classes, respectively a grade 6 class (18 pupils aged 11-12) and a grade 8 class (18 pupils aged 13-14), and in two grade 7 classes (20 pupils and 17 pupils aged 12-13). Each time the class teacher was present.

The two trainees presented the activity to pupils as a game for pairs, named "Discovering geometrical figures", in which one of the two components gave a list of "clues" which had to lead the other one to "identify" the figure. The pupils' work was structured as in the training course session. We needed to spend more time than we expected to explain how instructions had to be given (nevertheless we found non appropriate formulations in the written productions). The chosen figures were *rhombus*, *trapezium*, *isosceles triangle* and *cube*.

In more than one class, pupils wanted to experiment and try both roles within the pair. In one of the classes, the experimenter proposed a modification of the activity to avoid pupils helping each other within the pairs, through an exchange of forbidden information. The following modalities of work were followed: both pupils of each pair were given a sheet with the name of a figure to be secreted (the names of figures in the two sheets were different); each pupil had to write down a list of instructions so that their partner could draw a figure; successively the instruction sheets were exchanged and each pupil could read, comment in writing on the received instructions if they were not clear enough, and draw the figure. At the end they had to give back the sheet with instructions, comments and drawn figure to their partners.

At the end of the activity each pair commented on their work with their classmates and the teacher experimenter.

Collective discussion of feedback

Number of trainees: 18

Overall duration: two hours

Those who worked in the classrooms described their experience.

One point made was about the amazement pupils experienced when facing the proposed activity: it was unusual and they felt unprepared and feared they would be assessed negatively by the teacher. Only after being reassured that it was a game intended to help them study geometry could they feel more relaxed and free to express themselves.

Several pairs, especially in a grade 7 class, were influenced by the fact they "had to" provide a correct drawing of the figure, and they made it "appear" anyway, although it was incompatible with the instructions written on the sheet (a clear sign of illegal exchange of information). Due to this, the teacher-experimenter introduced a modification when he proposed the activity in the other grade 7 class, asking pupils to play the same role simultaneously, but on two different figures. Results were, as expected, more meaningful, and showed a generally good correspondence between the set of instructions and the drawn figure.

We also observed that pupils, (as did the trainees in the training session), evaluated the task of giving instructions as the most difficult one and admitted they had doubts about the way they expressed themselves. Those who received instructions admitted they were often in trouble and sometimes they made the drawing after interpreting instructions personally (*"If I had not understood what it was I would never get to draw the figure: some data were a bit* crazy*"*).

Some works produced by pupils were examined: they showed different levels of appropriation of the use of geometrical language and in some cases a strong contrast between what they wanted to describe and the way of doing it. Another aspect we pinpointed was the difficulty of giving a meaning to the expression "give a definition of the drawn figure" which for many pupils meant recognising it and writing down its name, and for others, listing some of its properties.

Comments

The proposed activity led trainees to reflect upon a number of aspects.

- Difficulty in using mathematical language correctly: uncertainties, doubts and mistakes shown in providing instructions express the need to favour, in teaching, the verbalisation process, which induces students not only to make their ideas explicit but also to try and make it in a clear and correct way to make themselves understood.
- The need to use the linguistic instrument appropriately, its use is a fundamental step towards the construction of knowledge, although it requires a considerable time for maturation.
- The need to develop activities like this one, because they offer information about pupils' knowledge, the conceptualisation level they have reached, possible gaps, and misconceptions. This information is fundamental to be able to intervene in the classroom with appropriate and well planned teaching actions.
- The more general need to develop geometrical discourse through a coordinated interaction of different registers (verbal, graphical and symbolic) and the acknowledgement of the important role played by perception and visualisation.

One trainee teacher wrote this comment:

> *Personally I believe that the activity we carried out was very interesting. Because it is not easy even for those who have a rather deep knowledge of the subject to convert spoken language into graphical language and vice versa. Given that, I think the same activity carried out in a lower secondary school class can raise interest and curiosity in both the teacher and his pupils.*

Third piloting (at IUFM of Paris) and conclusion

by Franco Favilli

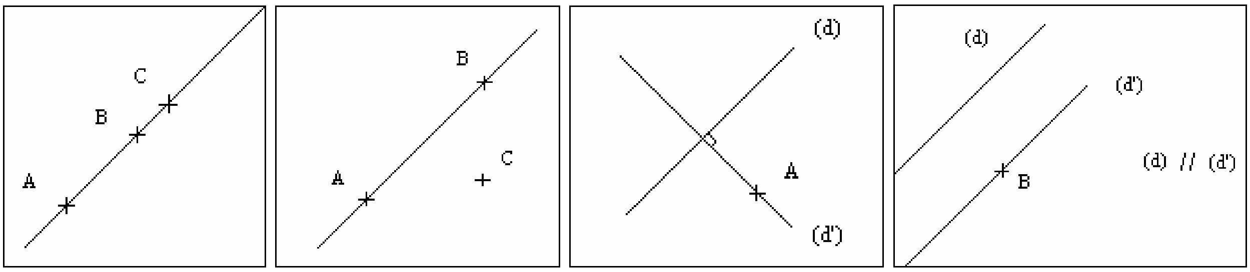
THE THIRD PILOTING

The proposal was also piloted, in a slightly different way, at the I.U.F.M. of PARIS by Catherine Taveau (class teacher: Cynthia Dobin). The class included 28 pupils, 11 to 12 years old, in their first year of secondary school.

The two principal aims of the teacher being the same as in the two previous piloting – to reinforce the pupils' knowledge of geometry language and to facilitate the transition from the appearance of a figure to its properties (i. e. from what can be seen to what can be known) – the proposal was implemented in two sessions.

First session

The receivers had only to draw the figure by hand, without geometric instruments**.**

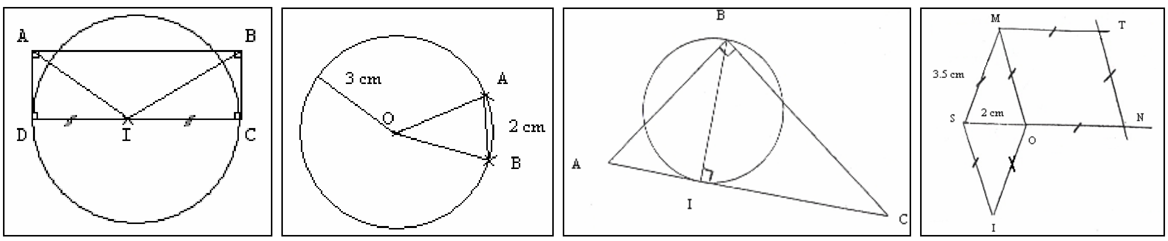


Picture 14. Drawing of lines

Comments – Many pupils tried to write instructions for drawing exactly the same figure (including for example measurements), and in several cases, the instructions were not precise enough, but this didn't prevent the receivers from drawing the figure correctly.

Second session

This session took place one month later. The teacher gave a geometrical mini-glossary to the pupils and asked the receivers to draw the figures precisely (with instruments).



Picture 15. Drawing of "circles"

Comments – The instructions for the first three figures were mostly correct, except for the last one. The teacher decided to extend the sessions with a computer activity, using Cabri Géomètre.

The teacher was very interested in this work, and became convinced of the difficulties for pupils of this age to use precise geometrical language. But it became obvious to her that such tasks (including those with Cabri) are good ways for pupils to understand the difference between describing and defining a figure.

CONCLUSION

To make pupils better aware about the difference between describing and defining a figure and to lead them towards the concept of definition of a figure, in the first piloting at the University of Pisa, it was decided by the trainees that went to the school:

- at first, to ask pupils to make a list of all the properties they could ‘see' in the given figure;
- then, to consider each of these properties and to compare it with the other ones;
- finally, to delete the property from the list if they thought it was a consequence of one of the others.

In this way pupils were convinced that the "surviving" properties represented a better, more refined and shorter description of the figure: something equivalent or very close to what their teacher called its "definition". The discussion in the whole class of the lists produced by the different groups of pupils, in particular the explanations provided to motivate the deletions, greatly contributed to make the introduction of one of the most complex and sophisticated mathematical topics, the concept of definition, both attractive and effective.

It is important to say that the three piloting proved how relevant for the learners (trainees and pupils) it is not only to master the geometric language but also to be able to organise and express an algorithmic process (the set of instructions)[18] to obtain the intended product (the drawing) from another learner who is just expected to execute the instructions. However, the difficulty (or impossibility) already mentioned for learners merely to carry out the instructions given, and the fact that their previous knowledge of the figure often pushed them to make the correct drawing, no matter what instructions were received, show how important it is to introduce the activity pointing out the importance of fully respecting the instructions received, in the way they are given and interpreted. For learners, especially for pupils, the wish to be right or, even, "to win" is too strong to resist any rule!

As regards the unbalance that could be seen in the tasks given to the pair (the instruction provider and the executor) strongly suggests to re-play the activity using a different figure and exchanging the roles in the pair, as it happened in the first two piloting.

[18] This ability cannot be taken for granted even within trainers and/or mathematics educators, as it was clearly proved by the workshop on this proposal that was organized as part of the scientific activities of the Congress SEMT '05 in Prague.

SUGGESTED READING

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Appendix B1: *Geometrical puzzles* – Learner's Guide

Materials for the activity:
it is possible to use only a sheet of paper, a pen and the form to be completed.

✓ Pupils work in pairs. Each pupil in a pair plays a specific role: one pupil (***the instructor***) gives instructions and the other one (***the drawing maker***) sketches a drawing, following the instructions received.

THE INSTRUCTOR

You are going to receive a piece of paper with the name of a geometrical, plane or solid figure, which cannot be shown or told to anybody.

Your goal is to allow your classmate to draw the given figure, step by step through a series of instructions.

1. If you like, you can draw the figure either before or during the game;
2. You can only give unitary instructions, i.e., each instruction should enable your classmate to draw just one single part of the geometrical figure. Let us explain it with a non-mathematical example: should in the sheet of paper be written *laid table*, the first related unitary (and therefore allowed) instructions could be:

   Lay down the table cloth – lay down a knife – lay down a fork – …

   while neither of the following instructions could be allowed:

   lay the table for lunch – we use it everyday for our meals.
3. Giving the instructions you can use some mathematical words such as *segment, axis, angle, etc.* but you can not use names of polygons (triangle, square, etc.),

THE DRAWING MAKER

Your classmate is going to receive a piece of paper with the name of a geometrical, plane or solid, figure. He/She will ask you to draw this figure through a series of instructions.

Your goal is to draw the given figure step by step.

1. Write down every instruction you are given by the instructor on the special form you have received, with your comments, if you have any.
2. Should an instruction be not clear, you can ask the instructor to say it again but not to explain it.
3. Should the instruction still be not clear, you can write it down and make your comments on the special form.
4. Execute the instruction and tell the instructor when you are ready.
5. You should not make corrections to the drawing. If you realize that you were wrong, write it on the special form.
6. When the last instruction has been executed, you should write the name, a description and the definition of the given figure.

geometrical formulas and names of objects that recall the geometrical figure to be drawn (for example, if the figure is a circle, you can not say "*Draw a wheel*").

4. Write every instruction on the special form you have received, with your comments, if any.
5. If the drawing maker asks you to repeat the instruction, you can do that only using the same words.
6. You can give the next instruction only when the drawing maker has executed the previous one.
7. When the last instruction has been given and executed, you should write down a description and a definition of the given figure.

The final drawing can be shown to the instructor and to the whole class only when all the pairs have finished.

Appendix B2: *Geometrical puzzles* – Instruction Sheet

Pair: ____

Instructions	Comments
1)	
2)	
...	

Name of the figure: ______________________________________

Description of the figure: __________________________________

Definition of the figure: ___________________________________

Appendix C: *Geometrical puzzles* – Teacher's Guide

The aim of this guide is to help teachers adapt the teaching activity to the competences of the class where it will be carried out. Since the activity is addressed to lower secondary school pupils, we have to consider that at this age (11 – 14 years) pupils' mind is just opening on to abstraction. Moving from the concrete to the abstract form is peculiar to this age.

Aims

The activity aims on one hand at strengthening pupil's ability to use the language of geometry and on the other hand, at developing, the teacher's ability to construct geometrical concepts in pupil's mind, without being too much constrained by definitions.

Requested background knowledge

The activity requires that pupils know a few basic geometrical concepts, such as segment, angle, perpendicular and parallel segments.

Complementary teaching materials

The geo-plane is a useful tool for the activity because it can represent a real image of a spatial context. Furthermore, it does not hinder pupil's free thinking.

Description and remarks

1. The teaching activity is presented to pupils as a *game*, thus decreasing their anxiety related to a possible assessment and allowing them to live the educational experience without any stress and actively. The activity should be introduced as *a role game*. There are two, quite different roles:

   - the pupil who gives instructions (*the instructor*);
   - the pupil who receives instructions (*the drawing maker*).

   The instructor is given a piece of paper with the name of a geometrical figure and he is supposed to lead the drawing maker to draw it.

2. The *speech* should fit pupils' age. Therefore, the teacher should not use the imperative form, that is authoritarian and recalls the language frequently used in the maths textbooks (e.g. *calculate the following expression*, *solve the following problem*), but should rather use the first plural person. However, if necessary, the imperative could be smoothened by the use of the interrogative form accompanied by the verb *can*.

3. For this type of activity pairs should not necessarily be homogeneous, because the objective is *communication*, both active and passive, between pupils. However, the teacher could form a homogeneous pair at least. In the first phase of the activity work in pairs is important to ease socialization.

4. As regards the *geometrical figure* to be drawn, the choice might be between a figure that pupils have already studied and a new one. On one hand, the background knowledge of the geometrical figure could reinforce pupil's knowledge of its properties and ease communication in the pair; on the other hand, it could activate in pupils pre-organized mental schemes. For example, the pupil in the pair who receives the instruction could, at a certain point, carry on with the drawing just because he/she understands which figure is to be drawn and not because of the instructions he/she is getting from his/her peer. Something similar could happen to the pupil who gives instructions, because he/she may hardly understand the different interpretations of his/her own instructions; for example, the instruction *draw two parallel sides* does not make it clear:

   a) whether the sides are congruent or not;

   b) which is the distance between the sides;

   c) whether the sides have an endpoint on a common perpendicular line or not.

   Very likely, the pupil who receives this instruction will draw two parallel sides of a square.

If we use a figure unknown to pupils, instead, they will be more careful in the way they give instructions, as well as in the way they execute them, because they do not have any related mental scheme yet. However, in this case, the activity might possibly be more difficult to pupils.

5. The use of squared sheets of paper can ease the task of both pupils in the pair (the one who gives and the one who receives instructions). However, this could represent a limit because it could suggest pupils' preferential routes (e.g. the instruction Draw an oblique segment could be followed by the drawing of a segment with 45° inclination, because of the squares in the sheet).

   In the case the teacher decides to use a blank sheet of paper, it would be useful to allow pupils to use both the ruler and the square.

   Also the pupil who gives instructions should be given a sheet of paper where he/she could draw the figure as well, for a visual support. In fact, at this age (11 to 14 years), pupils have poor abstraction ability and the drawing of the figure according to their own instructions could ease their own monitoring of the procedure.

6. As the activity is based on communication between pupils in each pair, the teacher should focus pupils' attention onto the fact that only single instructions are allowed. The notion of single or *unitary instruction* could be quite controversial: it is up to the teacher to make a choice and explain it to pupils. For example, in order to have the diagonals of a rhombus drawn, two different sets of instructions might be given:

   a) *draw a segment AB – call M its middle point – draw a segment MC perpendicular to AB – draw the segment MD congruent and adjacent to MC* (a sequence of four unitary instructions).

   b) *draw two perpendicular segments intersecting in their middle points* (just one non-unitary instruction).

   In the activity form given to pupils, it could be useful to provide an example of unitary instruction not from a mathematical context but from daily life, with the aim of making them feel free in developing the activity.

7. At the end of the activity pupils are asked:

   a) to write the name of the figure they have drawn;

   b) *to describe* it;

   c) *to define* it.

   This last phase of the activity is useful for pupils' construction of the concept of the given geometrical figure, through preliminary steps towards its definition. Furthermore, depending on the class context, the teacher should choose whether to ask for the definition of the geometrical figure or not.

8. The *final debate* in the class is an important phase of the activity because it allows the teacher and the whole class to observe the final drawings, to compare the set of instructions and the drawing sketched by each pair, to hear and discuss different ideas about the given figure. In view of that, the following strategies could be adopted:

   - at first, a pair should present and describe their activity to the classmates, that are invited to ask questions and make comments, thus creating the conditions for a real discussion among peers;

   - after that, a change in the pairs' composition could be suggested, thus having, for example, the instructor of a pair working with the drawing maker of a different one. This strategy should make pupils understand the importance of using a univocal mathematical language and terminology.

9. *Changes* to the activity

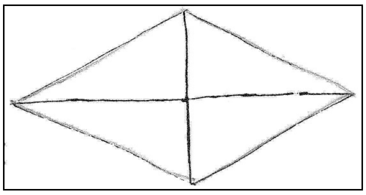
   - To provide all pupils in a class with the same set of instructions for the drawing of a given geometrical figure. Some of these instructions could be given in an ambiguous way, thus allowing for an investigation on pupils' reaction to different interpretations.

- To ask pupils to work on the drawing of a non-standard geometrical figure.
- To make different groups in the class. Each group choose a geometrical figure and set a sequence of unitary instructions to draw it. The teacher is asked by each group to be its drawing maker and draw the chosen figure.
- To make different groups in the class. Each group gives a set of unitary instructions for the drawing of a geometrical figure to another group and vice versa, as in a contest.

Appendix D: *Geometrical puzzles* – Two trainees worksheets

Example 1: Construction of a rhombus

List of instructions	Comments by the receiver
1. Draw a horizontal segment	
2. Take the midpoint of that segment	
3. Draw a vertical segment going through that midpoint	
4. Points of the segment must be equally distant from the segment's endpoints.	"*You can not understand which segments it refers to*"
5. The two segments must not have the same length	
6. Join the two segments' endpoints.	

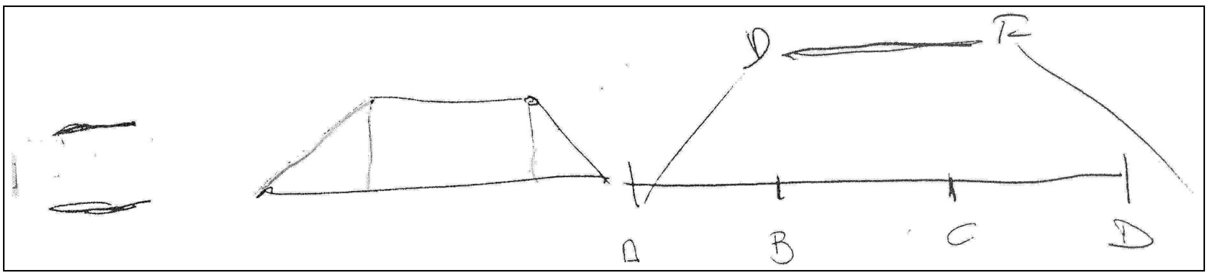


Actual drawing of the rhombus, as obtained from the above instructions.

Definition of the figure (from the instructions receiver) – "Rhombus: parallelogram with perpendicular and not equal diagonals."
Definition of the figure (from the instructions provider) – "Rhombus: plane geometrical figure which can be viewed as a particular case of a parallelogram with the opposite sides pair-wise parallel, the interior angles pair-wise equal and the perpendicular diagonals not having the same length."

Example 2: Construction of an isosceles trapezium

List of instructions	Comments of those who give instructions	Comments of those who receive instructions
1. Draw a horizontal segment		
2. Divide the horizontal segment into three equal parts		
3. Label the four points you got on the segment orderly A, B, C, D		"We were not told starting from the left"
4. Trace a segment parallel to BC	"Wasn't I clear? I skipped a passage: trace the perpendicular to AB by point B and the perpendicular to CD by point C"	"I can draw anywhere but I choose to draw it over BC and with equal length"
5. Label the segment EF		
6. Join A and E		
7. Join D and F		



Actual drawing of the trapezium, as obtained from the above instructions.

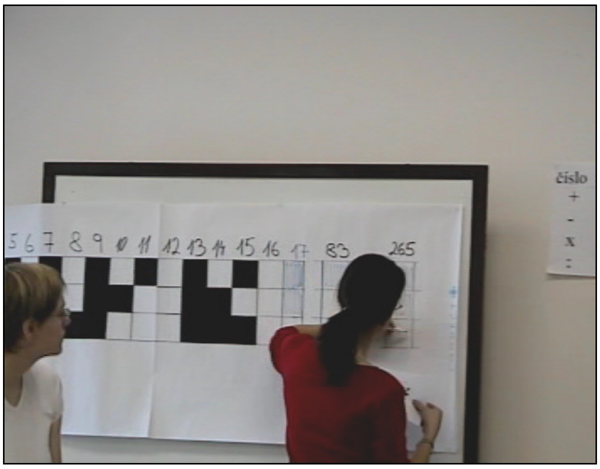
Definition of the figure: The figure is an isosceles trapezium i.e. a quadrilateral with two differently sized parallel sides and two equal sides. Interior opposite angles are supplementary.

“REAL WORLD” PROBLEMS

by Lucia Doretti*

INTRODUCTION

We consider it important that pre-service teachers master both mathematical problem solving and the choice and analysis of problems, together with the way of posing them in the classroom, so that pupils’ thinking processes may be better stimulated. The teacher must make several decisions about the organisation of their own teaching: these relate to the choice and systematisation of ‘good’ problems, the management of pupils’ personal solutions in the sharing phases (discussion), the possible ways for making these personal solutions evolve towards expert solutions, which are the main goal. In this context, a priori analysis becomes one of the professional tools helping teachers to formulate their choices and decisions (Charnay, 2003). The “Real world” problems proposal sits within a set of activities that stimulate work with problems starting from a suitable a priori analysis, in order to identify the mathematical concepts at stake and to determine whether, how and with what aims they can be used in teaching. The proposal provides an opportunity to select at least one of the suggested set of three problems “What a family!”[19], “Bizarre colouring”[20] and “The pursuit”[21], taken from the RMT – *Rallye Mathématique Transalpin* (Transalpine Mathematical Rally).



* Department of Mathematics and Computer Science, University of Siena, Italy.

[19] 12th RMT, Test I – Year 2004

[20] 12th RMT, Test II – Year 2004

[21] 11th RMT, Test I – Year 2003

Main piloting

by Lucia Doretti

THE PROPOSAL

The *"Real world" problems* proposal is the first of the two enacted within the LOSSTT-in-MATH Project, in the Tuscan SSIS (Specialisation School for Secondary Teaching), Siena Centre.

There is a teaching module about problem solving within SSIS. In this module trainee teachers are led to discover the central role played by the problem solving activity in pupils' mathematical education, through the experience of the *Rallye Mathématique Transalpin* (RMT). This is an international mathematical contest for both primary and secondary school classes, based on the solution of problems[22] (Jaquet, 1999), and some of the SSIS teacher trainers are directly involved in it, especially concerning aspects of educational research (Crociani et al., 2001; Crociani, Doretti, Salomone, in press). The choice of the RMT problems is due to their characteristics: to be relevant from the mathematical point of view and to allow connections with the mathematics program being taught in class, to motivate pupils and stimulate their involvement, to correspond with the various pupils' stages in cognitive development, to offer the possibility of a range of strategies for resolution and opportunities for developing the children's ways of representing the problem.

In the academic year 2004-05, the proposal involved first year SSIS students, Natural Sciences course, studying for a teaching qualification in Mathematics and Science in lower secondary school. The total number of trainee teachers was 15, none of which had a degree in Mathematics.

It was decided to work on the problem "What a family!" that requires logical-deductive abilities, more than specific mathematical knowledge. We think that such problems can find a good placing in the didactic practice, as they aim at the development of the pupils' reasoning skills.

What a family!

Mr and Mrs Calculations have 5 children, whose ages are different even numbers. The sum of the three daughters' ages is 30 years. The sum of the sons' ages is 14 years. The sum of the ages of the two oldest children is 26. The sum of the two youngest children's ages equals 10 years.

Indicate each child's age and state explicitly whether it is a boy or a girl.

Explain your reasoning and indicate all possible answers.

The text of a problem is presented to trainee teachers, working in groups. They are invited to discuss and analyse the problem, making hypotheses about representations,

[22] The TMR's goals are explicit: problem solving, interactive working, responsibility of the class group, explicit statement of procedures used to resolve the problem, justification of the solutions identified.

strategies and reasoning paths pupils might use, together with possible difficulties and mistakes. Later the problem is presented in some classes, where pupils solve it working in groups and compare the procedures they followed with the present trainee teacher. The activity ends in the training course, with reports of classroom experiences and analysis of all the pupils' written productions, commented upon by the trainee teachers in the light of the *a priori* analysis previously carried out.

Mathematical topics to be developed: Problem solving

Objectives

For trainers

- To stimulate the use of teaching practices that view problem solving as playing a central role in the construction of mathematical knowledge.
- To guide pre-service teachers in the a priori analysis of a problem and in the subsequent a posteriori analysis of pupils' written productions.
- To provide guidelines and feedback.

For pre-service teachers

- To reflect on the problem solving activity and on the role it plays in pupils' mathematical education.
- To work with problems that not only require a mere application of known procedures but rather personal search, development of strategies, formulation of conjectures, checks and validation.
- To analyse a priori a problem before it is used in the classroom in order to evaluate either the mathematical notions involved or those that should emerge, to foresee pupils' strategies, modes of representation, difficulties, obstacles and possible mistakes.
- To observe pupils when they are engaged in the solution of one problem, working in small groups.
- To reflect on the fundamental role played by cooperative and collaborative activities in pupils' social, affective and cognitive development.
- To develop the capacity of analysing pupils productions not only with relation to the notions they used, difficulties they met or mistakes they made, but also for the different levels of sense, consistency and argumentation of the answers given.
- To reflect on possible differences between predictions made through a priori analysis and results of the a posteriori analysis of pupils' productions.
- To use the information collected through both a priori and a posteriori analysis of problems to make hypotheses about specific teaching interventions in the classroom, if an actual need emerges.

For lower secondary school pupils

- To gain expertise with problems that constitute a personal challenge and that stimulate both interest in and motivation for mathematical activity.
- To develop a capacity of working in groups and to learn the basic rules of scientific debate: to express freely ideas, conjectures, reasoning paths; to have mutual exchanges, discuss, make decisions, check and validate …
- To learn new procedures and representations from exchanges with others.
- To develop metacognitive capacities through a reflection on thinking processes that guided the solving process and that allow for producing a justification of the answer and the reason for choosing that particular path.
- To stimulate the use of hypothetical-deductive reasoning.

Description of the activity

The activity developed along three phases. The first and the third phase, each lasting 5 hours, involved the 15 first year students of the SSIS, Mathematical and Experimental Sciences specialisation; the second phase, lasting 2 hours was carried out in lower secondary school and was managed by two of the SSIS students.

<u>Phase 1</u> (*in the training course*)

- Trainers discuss with pre-service teachers about the meaning of the term problem, about the different types of problems and their use in teaching.
- Pre-service teachers, working in small groups, receive the text of a problem taken from the TMR, solve it and carry out an a priori analysis on the basis of given tasks.
- Each group presents and comments upon their work to other groups and both the solving procedures and a priori analysis carried out are discussed
- A shared a priori analysis of the problem is written down.
- They plan an intervention in the classroom in which pupils are expected to work in small groups on the problem.
- Two trainee teachers are chosen and both of them will have to present, manage the activity in a class and collect materials produced by pupils.

<u>Phase 2</u> (*in secondary school*)

The trainee teacher actually present in the classroom:

- proposes and motivates the activity for pupils, who work in small groups
- observes the work of one of the groups during all phases of the problem solution
- manages the final discussion about solutions produced by the different groups
- collects pupils' written productions

<u>Phase 3</u> (*in the training course*)

- The two teacher-experimenters describe their experience in the classroom.

- Pre-service teachers, divided in small groups, analyse all the pupils’ productions collected and write down their remarks.
- Analyses of pupils’ productions are collectively discussed and they are compared with the a priori analysis of the problem.
- Trainers and trainee teachers reflect upon and discuss possible didactical interventions that may help pupils to become aware of the mistakes they have made and overcome their difficulties.

Tasks

a) *Tasks for trainee teachers*

- Which features of the problem can be highlighted, in comparison to classical textbook problems?
- What are the mathematical contents at stake?
- In which classes can the problem be posed?
- What mathematical notions might be mobilised in pupils?
- Which representation modes might be enacted?
- What strategies might be used?
- Will pupils be able to avail themselves of criteria to know whether they answered correctly or not?
- What might be pupils’ difficulties and possible mistakes?
- Plan a classroom intervention on a problem solving activity centred on the problem analysed, commenting upon the intervention a posteriori and proposing possible changes.
- Compare and comment on the solving procedures produced by pupils, and on how they were justified
- What type of didactical interventions might be designed to help pupils who met difficulties or produced wrong procedures?

b) *Tasks for pupils*

- Separate into small groups and carefully read the text of the problem.
- Exchange ideas and collaborate within the group.
- Understand both information and requests included in the text of the problem, choose a representation for the situation and identify a possible solving strategy.
- Check both choices made and results obtained.
- Reflect on the path followed and write down their reasoning.
- Exchange ideas and discuss collectively both procedures and representations used.

Piloting

a) In the training course

During the first hour, trainers discussed with pre-service teachers about the meaning of the term problem, on the different types of problems (problems for applications, problems to construct new knowledge, problems for the pleasure of searching and finding) and on their different possible uses in the teaching process.

After that we proposed the problem "What a family!". Trainee teachers worked in groups of two or three elements. Each group received the text of the problem together with the tasks of solving it and answering some questions in writing. These questions aimed at collecting information about mathematical knowledge, modes of representations, reasoning, strategies, difficulties and mistakes pupils were likely to encounter.

Trainers' aim was to lead trainees to reflect upon the important role played by an a priori analysis of the problem in view of better evaluating its didactical potential and possible uses in the classroom.

Each group was allocated one and a half hours to consider the problem, solve it and report on the solving procedure(s) they found on an overhead projector and complete their a priori analysis.

Various groups initially tried to solve the problem using algebraic tools, by setting up a system of equations. "Expert" processes were privileged as they were more familiar to pre-service teachers, although they were not suitable for managing the problem situation and in any case not generally usable in lower secondary school classes. Only after going back to a careful reading of the text could the groups unblock the situation and go ahead with hypothetical-deductive or combinatory reasoning starting from conditions provided in the text itself. Many were puzzled by the explicit request to indicate "all the possible answers", implicitly pointing to the existence of more solutions. Only one group was able to find the three possible correct answers, while other groups provided one or two answers.

The preliminary work of a priori analysis of the problem was carried out quickly and with synthetic answers by all groups. The general impression was that trainees did not acknowledge this activity as important, as compared to the solution of the problem to which they devoted all their energies. Someone also underlined their difficulties in answering some questions and in predicting pupils' behaviours.

The subsequent phase, lasting about one and a half hours, was devoted to sharing reflections and to a collective discussion. One member for each group illustrated the whole group work. Discussion centred on the different modalities used to solve the problem, on the incompleteness of reasoning paths that did not lead to finding all solutions and led to the identification of other different procedures. Trainees particularly highlighted the educational value of posing problems that have more than one solution, as opposed to the widely used model of problem necessarily having a unique solution.

The debate that followed led trainees to think about their a priori analyses and to reformulate a shared one that took into consideration what emerged in the collective discussion. This analysis is reported below.

Shared a priori analysis of the problem

Which conceptual domains are encountered by the problem?

Arithmetic – Logic – Combinatorics

In which class or classes may the problem be posed?

Grades 7 and 8

Are there remarks that can be made about the text? If so, specify them and indicate possible changes.

The text is clear because it is broken into simple sentences.

It should be read slowly and carefully because it contains many conditions.

What pieces of mathematical knowledge can pupils enact or consolidate? Which possible new pieces of knowledge might be required?

Capacity of managing many conditions simultaneously.

Capacity of developing hypothetical-deductive reasoning.

Combinatory capacities (identifying all pairs and triads of even numbers that add up to a given value).

What type of representations, procedures or strategies might pupils use, taking into account the background knowledge they are supposed to have?

To express data through symbols and schemes so that they may be better visualised and easily checked.

We believe that procedures are based either on hypothetical-deductive or combinatory reasoning (analogous to those found by trainees)[23]

What difficulties might pupils meet and/or what type of mistakes might they make?

Forgetting some conditions because there are too many to be taken under control at the same time.

Stopping at the first solution they find, or, if they guess that there are more, stopping at the second one, without checking for other possibilities.

Using algebraic procedures inappropriately [as trainees themselves did].

Highlight some educational values of the problem at stake.

The problem has more solutions and suits group-work, because it favours collaboration, exchanges and discussion among individuals.

---

[23] *Example of a solution*. Deduce from the given conditions that the middle child should be 8 years old (44 – 36). Then you get that the possible age of the youngest children are necessarily 4 and 6. Then you suppose that the middle child is male: then the other boy is 6 and the youngest child is a girl and is 4. The two oldest children are then girls and there are two possibilities for their age: 10 and 16 years old, or 12 and 14 years old. Suppose now that the middle child is a girl: then the two oldest ones are a girl and a boy and so are the youngest ones. Since a girl is 8, a sum of 22 for the other two girls is left. The only possibility is 22 = 6 + 16. Deduce that there are three possible solutions: F16 F10 M8 M6 F4; F16 M10 F8 F6 M4; F14 F12 M8 M6 F4.

Teachers and trainers then devoted one hour to prepare an intervention in the classroom, in which pupils were supposed to work on the problems organised either in pairs or in small groups, in order to favour both exchanges and discussion and to provide trainees with a certain number of written productions about the same problem.

The trainee teacher present in the classroom in the problem solving phase was supposed to observe pupils' work: for this reason it was agreed that they focused on one group only and took notes of the procedures they followed in the different phases.

An observation sheet, reported below, was prepared for this type of activity, with some questions that could guide observation.

Observation of the pupil's work

Phase of reading and understanding the problem

Who is reading? Is there any discussion during the reading phase? How long does the reading and understanding phase last? Are there pupils who participate actively and express their point of view? Is there a leader in the group?

Phase of solution

When it comes to the solution of the problem does the group hold together or break up? Are there exchanges in this phase of research? What kind of exchanges? Are there pupils who do not participate?

Phase of validation and control of found answers

Are solution(s) discussed within the group? How? Is there a control on the process? How do they come to a decision about the answer to be given? Is there still a leader in the group?

Phase of solution editing

How and why do they choose a pupil for editing the solution? Do those who are not writing control the process?

b) *In the classroom*

Two trainee teachers were involved in presenting the problem in two grade 7 classes, one with eighteen and the other with twenty pupils, aged 12-13. Each trainee was interacting with the pupils for the first time. During the activity, lasting overall about two hours, the class teacher was always present.

In order to motivate pupils trainees presented the problem as a mathematical game and a challenge for the class, inviting them to organise themselves into small groups. Each group was supposed to read the problem carefully, discuss it, solve it through a shared strategy and explain in writing the reasoning followed, being allocated 50 minutes. Work produced by each group would have been collectively discussed and commented upon in order to decide who met the challenge. While pupils were working, the trainee teacher present in the classroom focused on one group and observed and collected information about their way of going about the various phases of the problem solving activity. For this aim, specially designed questions were used.

The activity in class ended with a discussion on the different procedures used by the various groups and on the justifications produced. This led pupils to make explicit judgements on the efficacy and/or suitability of these procedures. At the end of the activity pupils’ written productions were collected to be later discussed and commented upon in the training course.

c) *In the training course*

The activity, globally lasting about five hours, was carried out again with the whole group of trainees. Those who went into the classroom told the story of their experiences to the group.

These stories brought to light different modalities of interaction within the two groups observed, which influenced the work and its success. They later reflected upon the importance of developing in pupils the capacity of working in groups, which involves being able to exchange ideas, to give one’s own contribution and accept those of others. It was also noted that this capacity is difficult to acquire, especially if it is not appropriately stimulated and regularly used.

After that the trainees again organised themselves into groups, were given the written productions of the problem collected in the classes. Each group examined pupils’ work with the task of writing down information about understanding of the problem, strategies adopted, mistakes made and difficulties met, but also on the explanations provided (for instance distinguishing between a complete and fully justified answer and a simple verification of the result found).

The subsequent phase involved sharing work, with the single groups presenting their own remarks and discussing them with others. In this phase, pre-service teachers participated in a particularly active and interested way. The a priori analysis and discussion of the problem was acknowledged to have fostered a more careful a posteriori analysis of the protocols: trainees stated they were led to put themselves in pupils’ shoes and try to interpret both their “way of reasoning” and their difficulties.

The concluding part was a synthesis of the remarks emerged throughout the activity.

Conclusions

The experimented practice forced trainee teachers to manage a mathematical problem solving activity on a “real” problem to be posed to a class.

The starting point was requiring them not to accept the problem uncritically but rather to try and evaluate *a priori* difficulties that pupils might have met in tackling it, to define what notions, representations and procedures were involved and predict pupils’ possible difficulties and mistakes.

We noticed that trainees moved from an initial perplexity and underestimation of the work requested to a progressively acquired awareness of its validity, especially when pupils’ written productions were analysed and discussed.

In the same way, they highlighted the importance of choosing to make pupils work in the classroom in small groups, in order to exploit the stimulus that a peer-to-peer exchange can offer to a discussion, and to a comparison and to exchanges of ideas and therefore to personal growth.

During the activity some trainees proposed to try and change some of the variables in the problem (numerical data, tasks, context,…) and study the effects of such changes on the problem and consequently on the possibility of using it. This idea was considered as an interesting route for developing this work.

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Web links[24]

Transalpine Mathematical Rally
[http://www.irdp.ch/rmt/]
[http://www.math.unipr.it/~rivista/RALLY/home.html]

[24] Active on December 2006.

Second piloting

by Marie Hofmannová* and Jarmila Novotná*

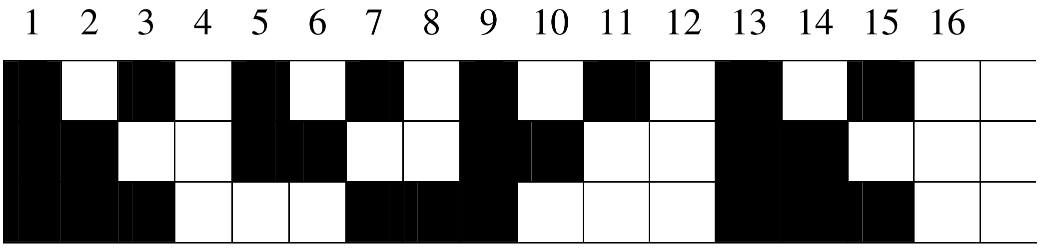
For the purposes of piloting LOSSTT-IN-MATH proposals, we selected such units that seemed to be compatible with our Content and Language Integrating Learning (CLIL) course content. The activity “Real World” Problems was selected from the set of proposals available within the project. We believe that word problems in general are more prone to the approach adopted in Prague, i.e. teaching mathematical content through a foreign language. The final selection was Czech students’ choice. The title is Bizarre Colouring.

PROPOSAL IMPLEMENTATION

Title: *Bizarre colouring*

Original text:

Maxime is filling in a square grid. In each line, the rule of colouring is different:



He has already filled correctly the first 15 columns. He states that the columns 1, 9 and 13 are fully filled. He continues with column 16.

Will column 83 be fully filled? And what about column 265?

Explain how you have found the solution.

Mathematical topics for development:

Solving word problems. Patterns. Combining arithmetic, algebra, geometry, combinatorics, etc.

Aims

For trainers

- Guiding the trainees from theory to practice.
- Guiding the trainees to adapt the lesson plan and teaching materials to the pupils’ age and level.

* Faculty of Education, Charles University in Prague, Czech Republic.

- Providing instructions and feedback.

For trainees

- Mathematics: Word problem solving, generalisation.
- Methodology: Material development to enhance pupils' motivation.
- Adapting a lesson plan.
- Trying and testing the student made materials that combine English and mathematics.
- Peer teaching.
- Classroom teaching.

For secondary school pupils

- Experiencing the teaching of mathematical content through the medium of English language.
- Building awareness of an imaginative and creative effort in the search of solution.
- Making conjectures, taking decisions, checking and verifying the results.

Piloting

a) *In the training course*

Charles University in Prague, Faculty of Education, an optional CLIL course, Mathematics taught through English as a foreign language.

10 teacher trainees, 22-25 years of age, 2 trainers, team teaching

Time table: 45-minute training session, 4 successive weeks

A priori analysis of the text

- Discussing the proposed problems from the perspective of possible mathematical solutions and language of the assignment.
- Choosing one of the three proposed problems as the basic problem for further elaboration (Bizarre colouring).

Preparing the lesson

- The trainers and trainees discuss in Czech how to best prepare the microteaching of peers. They assign roles and prepare the first draft of lesson plan.
- Peer-team teaching in English [*this stage was video recorded by one trainer*]: One stage of the proposed lesson is taught by two student teachers, the remaining trainees play the roles of pupils. One trainer takes notes on the blackboard for further discussion.
- Reflecting and analyzing (in English) the training lesson: Trainees present critical remarks both to the wording of the problem and the execution of the lesson plan. The necessity to change the assignment in order to fit real life is

emphasized. The trainees volunteer to prepare ~~a~~ new teaching material that would better correspond with the pupils' age and interests. For the result see Appendix E on page 200.

b) *In the classroom*

Secondary school in Prague, one 45-minute lesson taught instead of an English class, 14 pupils, 15-16 years of age, a teacher of mathematics, an English teacher, two teacher trainers, a teacher trainee – observer

Teaching the lesson [*This stage was video recorded by one of the trainers*]

- Introduction: The teachers organize an ice-breaking activity "Names scrabble", introducing each other. Teaching material: Square grid.
- Revision of mathematical terminology necessary for task completion.
- Teachers assign the original (RMT) version. Pupils solve it either individually or in pairs.
- Teachers change the focus of the lesson (from mathematics to English): They introduce Maxime, the character from the "Fashion World Magazine". They distribute the "Fashion World Magazine".
- English language: Teachers check pupils' listening and reading comprehension.
- Mathematics assignment: Pupils answer the questions from "Fashion World Magazine".
- Solutions are checked with the whole class.
- The teachers conclude the lesson.

c) *In the training course*

A posteriori analysis – reflecting on the lesson

The discussion was conducted in English. The items were:

- lesson analysis
- comments
- critical remarks
- suggestions for alternatives.

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Third piloting (at the Catholic University in Ružomberok, SK) and conclusion

by Lucia Doretti

The theme of the proposal, focused at problems and problem-solving, for its complexity and numerous implications is fit for being dealt with in a variety of ways, depending on the point of view that we want to adopt to make it emerge in the didactic practice and in teacher training.

The diversity of the viewpoints can be clearly seen in the report from the piloting by the different partners. For the partner who presented the proposal, the main focus was on the didactic situation. In the activity with the teacher trainees, who worked on the *What a family!* problem, the central role was assigned to the *a priori* problem analysis, completed by the subsequent *a posteriori* analysis of the pupils' written production. Teacher trainees had to analyze the problem before proposing it to the classes; and to confirm, to correct or to disprove, by the comparison of the experimental data, what they had foreseen *a priori*. The general lines of the activity and the way it was implemented let the *a priori* be perceived as one of the professional tools that can help the teacher and orientate him/her in the choices to be made and the decisions to be taken.

For the co-piloting partners the most important aspect in the proposal was the verbalization. In fact, the nature itself of the Content and Language Integrated Learning (CLIL) training course, within which the activity was implemented, required from the teacher trainees to deal with mathematical contents by the use of English as a foreign language. It was therefore necessary to make available a didactic situation that fitted for a suitable formulation in linguistic terms, both to motivate the learners to the use of English and to stimulate them on the mathematical side. During the preparation of the lesson the teacher trainees worked on the text of the *Bizarre colouring* problem, suitably modifying its context and assignments. The new version differs from the original one for fantasy and originality: the idea of a contest based on a mathematical quiz to get great discounts or a T-shirt free of charge makes the problematic situation concrete, making it closer to a situation from the real life. Trainers thus obtain good material to stimulate and enrich learners both on the linguistic side (by the use of English) and the mathematical side (in this version the concept of least common multiple is involved).

As regards the non-partner institution (the Catholic University in Ružomberok, Slovak Republic), the study of the proposal represented the opportunity to tackle, with the trainees, the issue of the complexity of the text of a mathematical problem. They started considering that a problematic situation can be seen as a structure that contains several inter-related parameters. The knowledge of what has been given has an effect on what has to be found and can make the solving activity differently complicate, at mathematical level, for the learners that have to develop it. In the implemented activity trainees were led to create a scaled series of word problems and to evaluate the difficulty level of the solving process, before piloting this kind of material in the classrooms, with the pupils.

Appendix E: *"Real world" problems* – "Fashion World Magazine"

FASHION WORLD MAGAZINE

SQUARES ARE IN

HIT OF THIS PRAGUE SPRING

Get superb discounts in our contest!

More information inside!

TO BE SQUARED = TO BE IN

It is a simple equation. If you want to be IN in the

coming spring season, put on a squared T-shirt. According to reputable fashion designers, the season will be full of squares. In this edition of the Fashion World Magazine, you can order a T-shirt with any squared patterns you can imagine. In addition to this, you can get **great discounts or a T-SHIRT GRATIS** ! Join our contest and win a

MAGNIFICENT T-shirt!

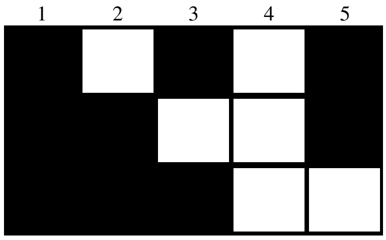
Don´t miss your chance and order a T-shirt. **Only now you can choose your own pattern!**

Follow fashion, don´t fall behind!

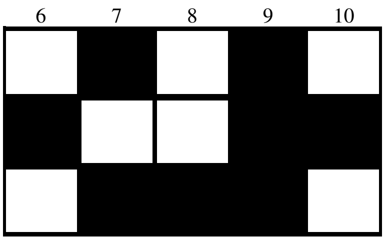
Advertising

These patterns were prepared by our designers only for you.

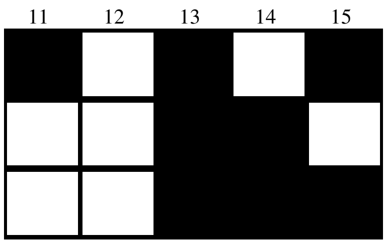
PATTERN No. 1



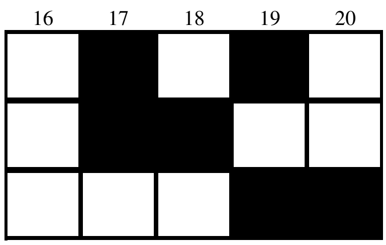
PATTERN No. 2



PATTERN No. 3



PATTERN No. 4



Show us your CREATIVITY and find **your own pattern.**

BUT DO NOT FORGET. **Your design must fit our offered SET!**

And now a SUPERB OPPORTUNITY ONLY FOR YOU!

Find the right answers to our quiz, fill in the information sheet, cut it off and send!

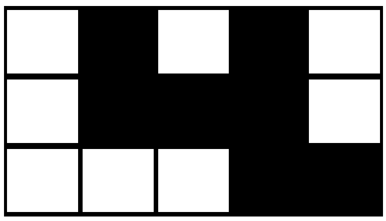
Everybody wins!

EACH CORRECT ANSWER = 10% DISCOUNT

Quiz

Question No. 1

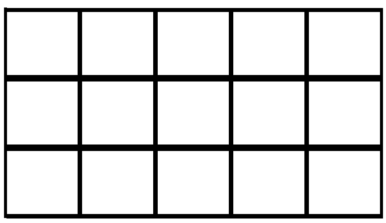
Is this a pattern from our set?



Question No. 2

If our catalogue contained all the other patterns, what would the pattern No. 16 be like? (Blacken the appropriate squares!)

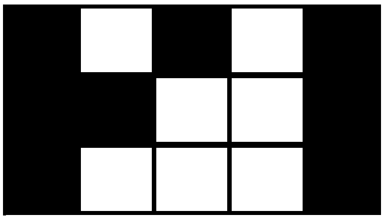
80 81 82 83 84



Question No. 3

What is the number of this pattern?
(If there are more possibilities, write ALL of them!)

? ? ? ? ?



Question No. 4

Choose a pattern from the set and write the appropriate number.

THOSE WHO SOLVE THE QUIZ QUICKLY
WILL GET A FREE T-SHIRT!!!

CUT OFF HERE -- CUT OFF HERE

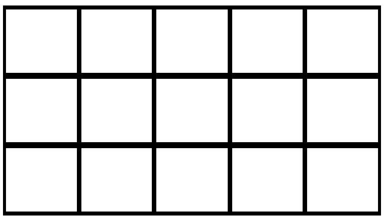
Name:
Surname:
Date of Birth:
Country:
Address:
Telephone number:
Your size:
Colour: Green Blue Red Grey Black
Pattern No.:

Answer No. 1:

Yes - No

Answer No. 2:

80 81 82 83 84

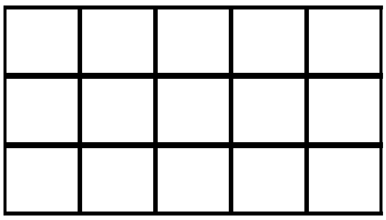


Answer No. 3:

It is the pattern number:___

Answer No. 4:

I have chosen this pattern:



Its number is: ___

GOOD LUCK!

TANGRAM IN MATHEMATICS FOR LOWER SECONDARY SCHOOL

by Jaroslava Brincková*, Miroslav Haviar* and Iveta Dzúriková**

INTRODUCTION

Learning is the outcome of an activity and it is also developed via activity. Among activities which pupils perform quite often are mathematical games. If such games are performed according to the rules satisfying certain didactical goals, they are called didactical games in the educational process. These didactical games include various geometrical puzzles, among them an old Chinese puzzle called Tangram. From the educational point of view, Tangram assists in teaching geometry via developing:

1. geometrical knowledge,
2. reasoning,
3. geometrical imagination.

Geometrical imagination is ability to sense:

- geometrical shapes,
- their size and position in space,
- a given shape in different space positions,
- changes of shapes in their size, structure, etc.,
- a shape in space according to its plane projection and a word description,
- a plane representation of a given shape in space.

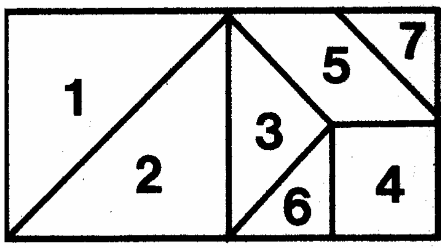
* Pedagogical Faculty, Matej Bel University, Banská Bystrica, Slovak Republic.

** 8. ročné evanjelické gymnázium, Banská Bystrica, Slovak Republic.

Main piloting

by Jaroslava Brincková and Iveta Dzúrikova

In teaching geometry, various activities can be carried out which strengthen the geometrical imagination by modelling via a paper-Tangram in E_2 (plane) or via a kit-Tangram in E_3 (space).



Picture 16. Tangram tiles

Rules for using Tangram

- All seven parts of Tangram must be used when creating any shape.
- No parts of Tangram can overlap.
- All parts can be used upside down if needed.

In teaching geometry, parts of a Tangram-kit can essentially be used in two ways:

- *To model a given prescribed shape* – here a constructive imagination, a sense for geometrical shapes and their properties can be well practiced; a child is sensing an area.
- *To fill in a bound area with parts* – here there are three possibilities:
  - The shape is given by its boundary.
  - All points of the shape are in one colour – a full shape.
  - The shape is placed in a square net.

When modelling via a given prescribed shape, the pupils have to compare the boundaries of the shapes, then to choose a corresponding part of Tangram and to assign it suitably oriented into the created shape. The pupils are imagining geometrical shapes**,** their size and position in space, the same shape in different space positions, etc.

When filling a bound area with parts, there are different levels of difficulty. As investigations show, the pupils of lowest grades do not sense the square net as a tool which assists them in their work with rectangles, but sense it as a two-coloured environment, thus as a paper with figures. Their have to learn gradually "to sense" the parallels and the normal. They are most successful in their work if the given shape is given by its boundary.

In teaching geometry at lower secondary school, Tangram can be used in various motivating tasks, in practicing areas, perimeters, axis symmetry, and similarities of shapes, in proving the Pythagoras theorem and in exposition of rational numbers. It also contributes to practicing the isometric transformations in geometry. However, it is not an ideal tool for teaching the geometrical concepts as it consists of only one of seven types of triangles (the right-angled isosceles triangle), from two quadrangles (the square and the parallelogram) and it does not contain a circle.

The main idea

The influence of one-coloured and multicoloured educational aid on the work efficiency of pupils. Developing an ability to sense a boundary and area of a non-convex plane shape.

The influence of a graphical environment (square paper, coloured paper, plane-white paper) on the ability to project a given model into plane-drawing. Find out the perimeter and area of the different parts of the puzzle.

1. **Title:** ***Tangram for measuring perimeter and area***
2. **Mathematical topics for development:**

Measuring planar shapes by using a non-standard measure unit

3. **Description of the activity**

The general aim of this proposal is to make the teacher trainees think of the importance that problem activities of measuring can bring to the mathematical development of pupils. We use the game of Tangram in seminars for the teacher trainees in their preparation for teaching geometry to pupils aged 11–14 (that is, at lower secondary school). The main goal is the development of creative thinking and geometric imagination of pupils. We also aim at preparing a school activity in which we deal with the concepts of perimeter and area in different contexts. We want to use Tangram to demonstrate isometric transformations in measuring perimeter and area.

We focus on the following partial aims:

- Didactic clarification of the sequence of steps in modelling the geometrical terms of perimeter and area of planar shapes*: perception – modelling – drawing in the plane – measuring – derivation of functional relations.*
- Describing van Hiele[25] levels of geometrical thinking, in particular with focus on deduction of functional relations using geometric terms.
- Modelling the world of numbers and shapes using a line segment as a unit.
- Finding relations between perimeter and area of different shapes.
- For 14 year pupils only: measuring the sizes of different shapes and calculating their perimeter and area by using also Pythagoras theorem or algebraic expressions.

[25] Van Hiele, P.M.: Structure & Insight. Kluwer Academic Publishers, Dordrecht, 1983

4. Aims

For pupils

- Combining the use of arithmetic, algebra and geometry within given activities.
- Using Tangram puzzles for modelling and measuring perimeter and area in plane geometry.
- Making conjectures, taking decisions, checking and verifying the results.

For teacher trainees

- In Mathematics: To study different problems of measuring in geometry. (To model relations between numbers and shapes).
- In Methodology: Working in groups – development of didactic material to enhance pupils' motivation. Testing of the materials carried out in 4 steps:
  - perception, modelling and drawing;
  - definition of concepts and measuring;
  - procedure for composition;
  - decomposition.

For trainers

- Guiding the trainees to adapt the lesson plan and teaching materials to the pupils' age, level, individual needs and responsibility for selection of tasks, etc.
- Providing instructions and feedback.

5. Assignments

For teacher trainees

Perception, modelling and drawing

<u>Task no 1</u> – The teacher trainees become familiar with the rules of the game Tangram. They draw pieces of the game according to Figure 1 on a paper. The teacher trainees prepare the game in two versions, plain and coloured, meaning that for the plain version they leave the geometrical shapes 1–7 blank and for the colour version they use different colours for the neighbouring shapes. They cut the pieces of the Tangram in both versions. The teacher trainees use the pieces of the Tangram in both versions separately to model the different Puzzle pictures in Picture 17.

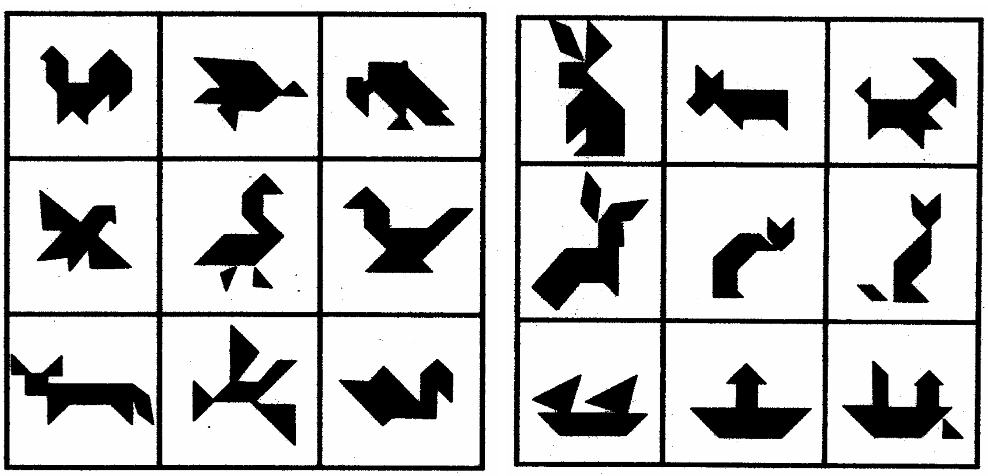
The teacher trainees use all the parts of the Tangram to create the different shapes in Figure 2 as well as other shapes, for example, a girl, a candle, etc.

They copy (draw by hand) each of the created models in both versions (plain vs. coloured) on three different sheets of paper: blank, squared and coloured.

The teacher trainees discuss the influences of the different background environments on the sheets of papers as well as different Tangram versions on the ability to copy the exact shape of the Puzzle pictures created by Tangrams.

Afterwards the teacher trainees discuss the influence of the different coloured pieces on the ability to perceive the outline of the shape. They should also note the different influence of the Tangram versions (plain vs. coloured) on seeing the boundary lines of the drawn pictures.

In the next step the teacher trainees discuss the potential of the Tangram game on teaching the classification of quadrangles to pupils aged 11–14.



Picture 17. Puzzle pictures

Definition of concepts and measuring

<u>*Task no. 2*</u> – (See also the Map of Terms on page **Errore. Il segnalibro non è definito.** or the Slovak website http://www.zoznam.sk/katalogy/Vzdelavanie/Slovniky/). The teacher trainees find the meaning of the concepts perimeter and area in different contexts in a thesaurus. They investigate the concepts perimeter and area in different contexts (geography, literature, electrotechnics, civics, arts, geometry…). By this activity we want to emphasize that the perimeter (the area) is in mathematics understood as *length of a closed curve* given by the (ordered) pair [number; measure] and not as a boundary (area) of a planar shape.

Procedure of composition

We can use two units – one unit is the side of the square 4 (call it s), the other unit is the hypotenuse of the triangle 7 (call it *h*).

We show that, given a *perimeter* (*area*), one can make, according to the instructions, planar shapes with different area (perimeter).

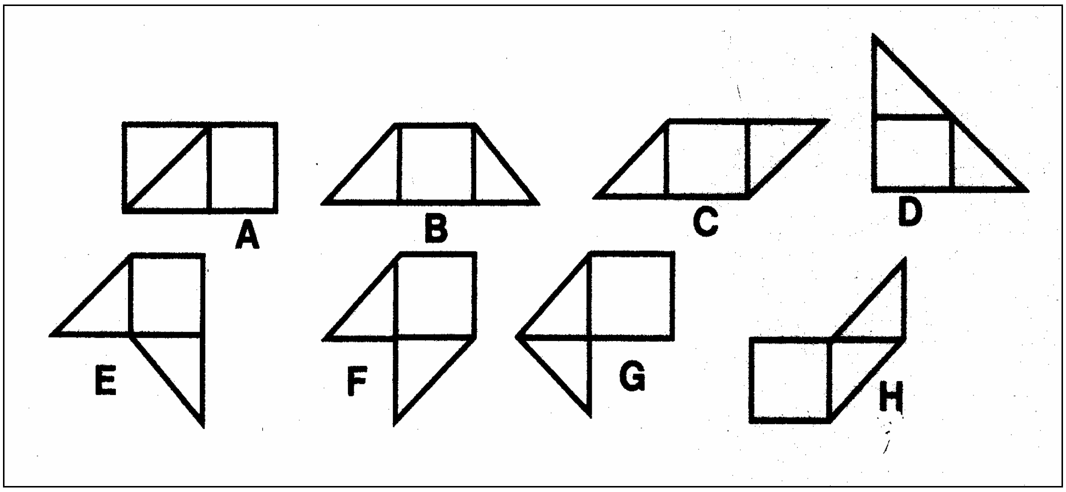
<u>*Task no. 3*</u> – From two congruent triangles of Tangram (i.e. from either 1 and 2, or 6 and 7), model planar shapes so that you identify the sides of the same length. Draw the modelled solutions in your exercise book. Express the perimeter of the modelled shapes using the length units *s* and *h*.

<u>*Task no.4.*</u> – From the square 4 and the two triangles 6 and 7 of a Tangram, make planar shapes so that you identify the sides of the same length. Find all solutions and classify them according to the perimeter, according to the number and the size of the angles and according to the parallel sides.

By fitting the shapes together, pupils can see that one side of the triangle is longer than the side of the square. So, there are possibilities for an interesting and didactically fruitful discussion – what do we do about this? Suppose we are not allowed to measure – how do we classify the shapes? Which shapes have the same perimeter?

We can use two units – s and h**.** So, the perimeters are: A is $6s$, B is $4s + 2h$, C is $4s + 2h$, … etc. (In fact, you can see that they are all $4s+2h$, except shape A.) This motivates the use of symbols (s, h) to solve a problem and also leads to the question

What are the perimeters of all the other Tangram shapes?



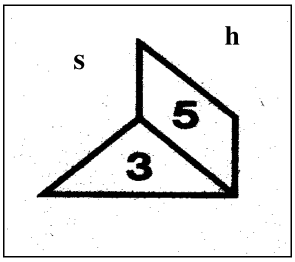
Picture 18. Results

Decomposition procedure

<u>*Task no.5*</u> – Students put together all parts of the Tangram and create: a) triangle, b) square, c) rectangle. Watch carefully and find differences between plain and coloured puzzles.

<u>*Task no.6*</u> – Create triangles from 2, 3, 4, 5, 6 and all parts of the coloured Tangram. Draw the colour models. Find all solutions consisting of five parts.

<u>*Task no.7*</u> – A little girl Barbara created a five-sided figure. Look at the Picture 19 and form a new one from the parts numbered 3 and 5. What other parts of the Tangram do you need to create the same shape? One of the solutions is to use the parts 4, 6, 7. Find all other solutions.



Picture 19. A five-sided figure

Area and perimeter of planar shapes

Task no.8 – Create all possible shapes from the triangles 6 and 7. If the unit for the Tangram pieces is the length side of the square **s** and the hypotenuse of the triangles is *h*, study the relations between the perimeter and area.

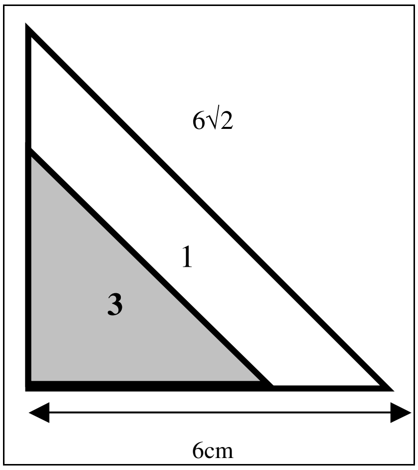
You do not have to know of height triangles nor do you need to measure the area in order to classify the shapes. We can use one unit of area – *T* (the area of the triangle 6 or 7). All shapes have the same area – *2T*.

Task no.9 – Create the shapes in Figure no.3 from the triangles 6 and 7 and the square 4 of the Tangram. Compare the perimeters and the areas of them.

Task no.10 – If the unit of the area is the one of the smallest triangle of the Tangram – T, find the areas of the different parts of the puzzle.

Special Task

A boy John put the middle triangle numbered 3 on the top of the large triangle of Tangram numbered 1 as seen in Figure no. 4. Calculate the area of the newly created trapezoid (coloured in blue) using the units *s* and ***h*.** (*Should it be the same as the area of the triangle 3?*) Express that area in cm^2: the length of the small side of the triangle no. 1 is 6 cm and the length of the hypotenuse is $6\sqrt{2}$ cm.



Picture 20. Triangle

For pupils

The pupils draw pieces of the game according to Picture 16 on a paper. They prepare the game in two versions, *plain* and *coloured*, meaning that for the plain version they leave the geometrical shapes 1–7 blank and for the colour version they use different colours for the neighbouring shapes. They cut the pieces of Tangram in both versions. The pupils use the pieces of the Tangram in both versions (plain and coloured) separately to model the different *Puzzle pictures* in Picture 17 and to become familiar with the rules of the Tangram game.

They copy (draw by hand) each of the created models in both versions (plain and coloured) on three different sheets of paper: blank, squared and coloured. Afterwards the pupils discuss the influences of the different background environments on the sheets of papers as well as different Tangram versions on the ability to copy the exact shape of the Puzzle pictures created by Tangram.

The pupils discuss the influence of the different background environments of the drawn pictures on the ability to see their boundary lines. They should also note the different influence of the Tangram version (plain vs. coloured) on seeing the boundary lines of the drawn pictures.

In the next step the pupils create a cat, a dog, a hare using all parts of the Tangram and discuss the potential of the Tangram game on teaching the classification of quadrangles.

They learn the mathematical concepts in English: base, height, hypotenuse, right angle, perpendicular and (in the case of the Tangram) isosceles, and the notions related to symmetries like transformation, rotation and translation.

They explain the terms perimeter and area in different contexts.

They can use two units – one unit is the side of the square 4 (call it s), the other unit is the hypotenuse of the triangle 7 (call it h). They find out that, given a *perimeter* (*area*), one can model, according to the instructions, planar shapes with different area (perimeter).

The pupils create models from the Tasks no. 3 and 4. Create all the shapes you can by putting congruent sides together. They discuss in groups how many solutions are there to this problem.

If the unit of the area is the smallest triangle of the Tangram – T, find out the area of the different parts of the puzzle.

Making conjectures, taking decisions, checking and verifying the results.

Bonuses in individual work for best pupils are the tasks no. 6, 7 and special.

For the trainers

- Guiding the trainees to adapt the lesson plan and teaching materials to the pupils' age, level, individual needs, responsibility for selection of tasks, etc.
- Providing instruction and feedback

Conclusion

This proposal is designed for teacher trainees of Mathematics in 6-9th grades at basic school (11-15 age) or in early grades of the grammar school and also as a compulsory part of the course in Didactics of Mathematics.

The venue: Pedagogical Faculty, Matej Bel University, Banská Bystrica.

The trainers: Teaching team created by university lecturers, 1 trainer and 2 teachers of Mathematics and 1 of English.

Trainees: 18 prospective teachers in the course Didactics of Mathematics.

The time schedule – 2 lessons per week

Week	Activities	
1.	Students	prepare Tangram – plain and coloured know and use the rules to work with the Tangram puzzle clarify geometric terms using of the Map of Terms
	Homework	motivation for classifying quadrangles use internet for your study work in pairs to plan lesson
2.	Students	discuss different solving procedures in pairs and groups present differences on the colour board form maps of the language terms use correct terminology in different school subjects (Slovak, Physics, Art, Science, Games, etc.) form critical analysis of presentations of lesson plans
	Homework	finish the lesson plan using interdisciplinary relations form analysis of teaching aims write teaching steps and tasks for pupils in the lesson plan
3.	Students	check your lesson plan prepare final discussion about lesson plan sequences preparation of 2 trainees who will teach in real class other students comment and check + prepare video recording
	Homework	analysis of the planned sequences prepare a lesson for pupils who did not understand the teachers materials
4.	Students	students and teacher watch video record and analyse the lesson concentrating on communication between teacher and pupil trainer classify trainees and comment on their creative work
	Homework	create your own logo using a Tangram for the course Didactics of Mathematics

Table 8. Time schedule

Realisation of proposal sequences

Realisation in the classroom

Evangelical Gymnasium Banská Bystrica, Skuteckého 5. It comprises 8 grades of lower and upper secondary school, class quarts, age of pupils 12/13, number of pupils

21. Mathematics in English, Geometry in English. Two teachers – English and Mathematics.

Teachers taught alternately. Student of course recorded a video.

Primary School Amos in Martin, Východná, class 5th, alternating teaching of Maths and Science. Number of pupils 23. Two teachers – teacher and trainee. One teacher taught. The teacher trainees recorded a video.

The classroom

Modelling in plane (E_2) – Teacher motivates the pupils.

Classification of quadrangles.

Composition and decomposition procedures.

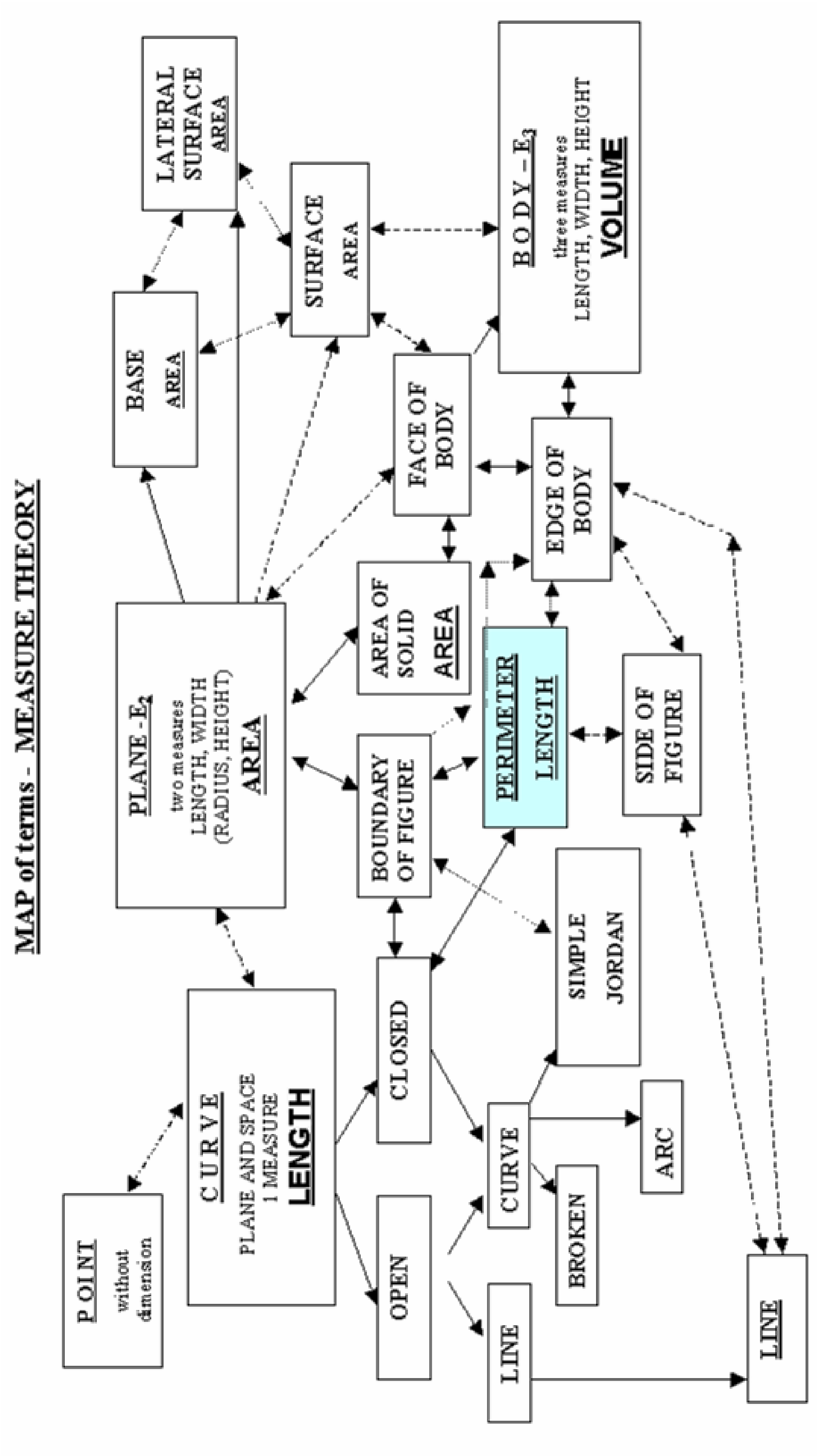
Perimeter and area.

SUGGESTED READING

Brincková, J. (1996) *Didaktická hra v geometrii*. (Didactical games in geometry). Bratislava: DONY

Brincková, J. (2001) *Tvorivé dielne 2* (Zamerané na didaktické hry). Banská Bystrica: PFUMB

Millington, J.: *Tangram. Puzzle picture to make you think*!


MAP of terms - MEASURE THEORY
P OI NT
without dimension
C U R V E
PLANE AND SPACE
1 MEASURE
LENGTH
PLANE - E_2
two measures
LENGTH, WIDTH
(RADIUS, HEIGHT)
AREA
BASE
AREA
LATERAL SURFACE
AREA
SURFACE
AREA
B O D Y – E_3
three measures
LENGTH, WIDTH, HEIGHT
VOLUME
FACE OF BODY
AREA OF SOLID
AREA
EDGE OF BODY
BOUNDARY OF FIGURE
PERIMETER
LENGTH
SIDE OF FIGURE
CLOSED
OPEN
SIMPLE
JORDAN
CURVE
ARC
BROKEN
LINE
LINE

Second piloting

by Brunetto Piochi*

Trainees have to arrange as many different plane drawings or plane geometrical figures as they can, by using a classical 7-pieces Tangram (eventually built up by themselves). Then they will have to look at geometrical properties (convexity, number of vertices …) of such different shapes, in order to discover general relations or to give a classification of them. In particular they are asked to use only some designated pieces, while trying to construct regular polygons. Area and perimeter properties of the (non congruent) figures so built have to be also considered.

Similar activities will be performed with pupils and the results of their piloting will be afterwards discussed with trainees.

Mathematical topics

The proposal is related to geometrical properties of drawings, in particular to the measures of area and perimeter and to isometric transformations.

Aims

For trainers

- Guiding trainees from theory to practice.
- Letting the trainees experience an activity on their own, before proposing it to the pupils.
- Providing instructions and feedback.

For trainees

- Discussing about basic notions of geometry and how to present them.
- Realizing the difficulty of defining and naming a “geometrical figure”.
- Experiencing a classification activity on non standard figures.

For secondary school pupils

- Knowing basic names and notions about some common polygons.
- Being able to measure the length of a segment (directly or, if necessary, by means of Pythagoras’ Theorem).
- Realizing the equivalence of plane figures which can be decomposed into the same parts.
- Working on plane figures by means of isometric transformations and their compositions, realizing that new figures are congruent to the previous ones.

* Department of Mathematics, University of Florence, Italy.

Description of the activity

Activities took place in SSIS School and involved 42 students, both first and second year, of SSIS, specialisation Natural Sciences, for the qualification to teach Mathematics and Science in lower secondary school.

Phases and timing

- Presentation of Tangram and activities about geometrical figures (1h30')
- Discussion and design of a proposal to be carried out in class (45')
- Piloting in the classroom (between 3 and 5 hours, depending on the classes)
- Final discussion (30')

SSIS students were given cardboard copies of a Tangram to be cut out. The three following activities were proposed and then commented upon jointly:

- To make up an 8 by 8 squares grid on which to report coordinates of vertices that had to be joined to form the sides of the figures that constituted the Tangram puzzle: (8,0) and (0,8); (0,0) and (4,4); (8,4) and (4,8); (2,6) and (4,8); (6,2) and (6,6); (4,4) and (6,6).
- To form all possible geometrical shapes using the square and the two small triangles and putting congruent sides aside. The obtained figures had to be classified according to the number of vertices, area and perimeter.
- To use all Tangram pieces to construct a known polygon: triangle, square, rectangle.

In the discussion that followed the activities, students were invited to answer the following questions, mainly focusing on the didactical aspects of the activity:

- Which competencies are involved in this type of activities? Which pre-requisites are needed? What type of learning is promoted?
- What difficulties did you meet in this activity? Do you think pupils would meet further difficulties? How can they be helped overcome them?
- How might we outline a classroom activity with this instrument? At what teaching level? Which are the most important points to focus on, in your opinion?

Later, two trainees who were already teaching carried out a classroom teaching experiment: the choice was due to the fact that they could work in known classes and include the activity in the classroom standard syllabus. The proposal, mainly sketched during the preliminary discussion, was taken by trainees and adapted to their own teaching context; experimentation took place in four classes (totally about 80 pupils aged between 11 and 14); one of these two classes used it as a peer tutoring activity with primary school classes.

Pupils were given a classic square Tangram made of 7 pieces (either to cut out or to construct on a coordinates grid). They were invited to construct different plane figures (either fancy or geometrical ones) by using these pieces, and then make hypotheses and verify conjectures about them. In particular they were invited to construct figures with some given pieces (in some cases all of them), identifying which figures are congruent, making conjectures about possible classifications and reflecting upon both extension and perimeter of the figures constructed in this way.

At the end, experimenters reported on their activity to others, making comments on the hypotheses emerged from the preliminary discussion. Finally the whole group proposed some particularly meaningful activities for further analysis.

PRESENTATION

Geometry learning, especially in primary and lower secondary school, is particularly important, beyond the mere set of learned notions related to a specific theme.

Geometry plays a crucial role in the formation of rational thinking both as *spatial organizer* and as providing a *rational description* of space.

“Geometrizing” our experience of the world around us is a primary mathematical activity, preceding the very counting activity. Children mainly and spontaneously tend to represent their experiences through graphical-pictorial activities, before enumerating objects around them. This graphical-pictorial activity tends to both represent and interpret our experience of reality; it is mathematics that at some point, offers specific instruments to describe these real objects: lines, points, figures…

Geometry thus originates from observation, manipulation, construction and representation of simple objects, from folding, cutting, putting together, from looking at both oneself and everything around in the mirror… The subsequent “geometrization” is neither easy nor simple, it requires a capacity of “interpretation” that allows for detachment from a naive view to get to a complex rational understanding. Geometrical thinking shapes up throughout the whole school life span through the different teaching and learning levels, merging together concrete and rational aspects of geometry, even though the former or the latter prevail in different moments of the school experience.

In order to exemplify this point let us consider “geometrical figures”. The first approach is to operate on (elementary and regular) geometrical figures, describing their shape and properties: this can be defined the “visual” level. This approach normally characterises the first years of primary school (6 to 8 years old pupils). Later we move to recognizing and describing figures on the basis of learned properties, at a “descriptive analytical” level. Then pupils construct definitions, look for characterising properties and need to argue and prove: this is the highest and most abstract level, leading to the “formal” one envisaging proof of theorems and study of geometry axiomatic system (or rather axiomatic systems).

Being able to operate with figures and drawing them become important instruments for geometry learning: drawing figures one is able to visualise features and properties, because properties of a geometrical object are translated graphically by means of spatial relationships. However, the inverse path, leading from the drawing to the geometrical object, comes from a human subject’s interpretation act: recognising visual spatial properties attached to geometrical properties is not a spontaneous activity and thus needs a proper learning process. A (geometrical) drawing can be interpreted in many ways in different contexts and perception comes in when an interpretation is constructed: this might be incorrect, especially when the reader's theoretical notions are limited and do not allow him/her to go beyond a perceptive reading.

Moving from the object to the geometrical drawing by identifying features and from the drawing to the geometrical object by means of interpretations show how graphical activity and its gradual refinement are both consequence and source of learning. For instance through them it is possible to highlight contradictions in theoretical misconceptions (it is extremely difficult to make the "heights" of a triangle that follows the squares borders meet exactly…) or advantages of a theory that permits us to "predict" general consequences (equality of the third sides of two triangles having two sides and the angle between them respectively equal…).

Highly relevant within this conception of geometry learning are those activities that are posed as borderline experiences, presenting both playful and graphical aspects and at the same time, offering opportunities for an abstract mathematisation. Too often does the teaching practice skip these transitional moments and the most delicate time is exactly the beginning of lower secondary school when insistence on definitions and formulae detached from a concrete context contributes to a (often definitive) distortion of the view of mathematics. Geometrical aspect is seen and perceived as defined by the mnemonic knowledge of definitions and formulae. Therefore it seems necessary to put enough care to the description and evaluation of this type of activity in teacher training.

The proposal about the use of Tangram we piloted lies on this educational line.

ACTIVITIES WITH TRAINEES

All SSIS students were given a copy of a Tangram puzzle on an easy-to-cut-out sheet[26].

In the introductory part of the experience we detached from the scheme proposed by Slovak colleagues: due to time limitations we avoided that SSIS students worked on the construction of free figures with Tangram. Trainees were anyway made aware that in this activity, as well as in any other manipulative or laboratory-like activity, the initial phase should involve free exploration; hence the need to leave pupils some time to "play", to explore the different pieces available and try to use them for various creative productions.

We then started a path with SSIS students by presenting on transparencies a Tangram puzzle and a set of figures that could be made with it. It was pointed out that it would be desirable that pupils be encouraged to work either individually or in small groups to construct these images or invent new ones. This phase might initially seem to be useless from a mathematical viewpoint (many trainees shared this opinion, although

[26] Trainees were also given, among other things, the scheme of exercises set up by Banska Bristica (SK) colleagues, of course translated, so that each trainee could have all material for discussion available.

they later changed their mind) but it is crucial on the motivational plane and for allowing pupils to make contact with the material and explore intuitively its limits and potentials.

Looking at the slides SSIS students were asked whether they considered the activity easy for their students and how they could think about a mathematical educational aspect of this phase of the activity (always keeping in mind the playful and motivational aspects). In the discussion that followed there were few and all agreeing interventions: the activity was easy, relevant to interdisciplinary links (with Arts and Technical Education) but scarcely meaningful from a mathematical viewpoint. We believe that this perplexity shows the distorted view of the discipline we referred to earlier: despite previous laboratory-like experiences, trainees hardly see the potential geometrical learning in activities where an informal approach prevails[27.]

It is interesting to show that later, in the classroom activity, this phase was dealt with before the actual geometrical work. In the final discussion they remarked that pupils found the activity relatively easy and could realize the proposed figures smoothly. However trainees themselves pointed out how this activity contributed to highlighting a series of “properties” of figures that teachers tends to take for granted, especially those properties that relate to the dynamic nature of the position figures take up in the plane (it is known that many pupils tend to visualise geometrical figures in a static way), or to the different configurations of the borderline between Tangram shapes, that come to constitute the different regions of the composed figure: this borderline may be a point or a segment, may include part or the whole side of an elementary shape etc. Working in class on finding a linguistic definition of these situations led to enrichment in the geometric vocabulary and served as a basis for next phase. In the light of this remark it might be not useful to include this phase also in the SSIS training activities.

Activity 1 of course requires notions about the Cartesian plane (together with precision and manual ability). Trainees met no particular difficulties with it but they supposed their pupils would meet some, since they did not have all the needed pre-requisites about the Cartesian plane. In case these pre-requisites were held by pupils, trainees proposed that pupils might be showed the Tangram shape and then provided with only some of the coordinates. In the discussion it was pointed out that, due to symmetry reasons and to the size $8=2^3$ of the chosen grid, all vertices of the Tangram portions on the grid have anyway integer coordinates, although many segments have irrational length. From the didactical viewpoint it was interesting to notice that different competencies are required to join points with given coordinates or to indicate coordinates of points in the plane: in this way it is possible to construct different requests, suitable to re-enforce or promote different competencies depending on the pupil’s needs.

Activity 2 presented to trainees two different types of difficulties (surprising for students themselves…): the need to identify and define a mechanism for

[27] We remind that trainee teachers involved in this activity are mainly Science graduates but not Mathematics graduates. Often their relationship with and conception of mathematics do not differ much to their pupils’ ones.

classification that identifies two congruent figures and the impossibility of "naming" all the figures. This latter point in particular highlighted the belief that geometrizing is too often synonymous with "naming". Certainly the activity was useful for pre-service teacher training, because, once in the classroom, they could manage pupils' "discovery" of non standard polygons with greater ease. Students also pointed out the relevance for teaching of this proposal in order to promote creative competencies in pupils, who are invited to experience a form of autonomous mathematical classification. However we decided to present the activity as group activity, since requested competencies are possibly not available to every single pupil at the age we considered: group activity will permit exchanges with all the possible advantages coming from this.

A reflection that came out immediately with trainees, but that might not be as such for pupils (and in fact it was not) is that it is not possible to classify by the area of figures constructed with the same pieces, given that these figures are equivalent, being all equally decomposable.

Difficulties met in activity 3 may be mainly reduced to difficulties (well-known in *gestalt* psychology) in de-structuring and re-structuring one's vision, so as to visualise the given figure as part of another one, whose structure in mind is again strong and rigid. We notice here that in classroom activities pupils were much faster and capable in carrying out these activities, possibly due to a less rigid structure of the geometrical figures they held. This was predicted by most SSIS students, who had supposed that pupils could be more capable because of an envisaged higher visual capacity.

CLASSROOM EXPERIMENTATIONS

Four of the SSIS students volunteered to present the activity in their classes. The scheme of the proposal was agreed during a collective discussion, adapting it to the different classes and to the topics of the syllabus they were working on. Trainees involved in the experimentation (the class teacher and another trainee) were asked to pay attention to the points highlighted in the discussion, also to test the hypotheses made about difficulties and meaningfulness of the activity.

A point shared by all experimentations was (also due to the fact that activities were carried out in February) that all the involved classes counted few pupils because of flu or winter extra-school activities.

The following are excerpts from trainees' final reports

Grade 6, 5 hours work, 12 pupils involved

[Tangram was made starting from the coordinate grid, the construction of which allowed for a revision of notions about the Cartesian plane. After that the teacher left pupils free to play with pieces]. As soon as they cut out the seven pieces they starter to compose them, turn then around, put them together to obtain images in such an enthusiastic way I did not really anticipate. The most amazing comment was "This is real mathematics!" which meant, as its author later explained, that "we are having fun and thinking a lot and racking our brains at the same time".

Then I proposed that rules were fixed to hold for everyone: we had not to overlap pieces, put them aside by the sides, and always use ALL the pieces. After an initial moment of doubt, pieces started to turn around every desk. A moment of glory was reserved to the Chinese girl who joined the class two weeks ago, without knowing any Italian, who, after copying and cutting the scheme in silence started to compose more and more complicated figures, laughing: the first woman, the first boat… I could not and neither wanted to interrupt them until a boy constructed "a trapezium, mister, a trapezium!". I took the opportunity: "Oh, yes, and it seems to me that we can construct also squares, triangles, rectangles…". It was a new challenge: always with all seven pieces. Almost all pupils engaged in the search and the rectangle that we, as SSIS students found in 5 to 6 minutes came out in less than a minute and a half. I checked discretely and invited the girl who made it to cover it, because I wanted to see what the others were doing: well, in less than 5 minutes every single student had their own rectangle constructed.

Starting from there I asked them to observe and reflect on the extension of each figure, and starting from figures that occupy the same surface we moved to think about them as made of the same pieces, which, although being moved around staying the same, give rise to different figures that nevertheless occupy the same surface. […] Pupils like this part very much because anyone could manipulate and compare pieces as they liked, get it wrong and try again. Other reflections came up when they had to observe the figures boundaries, after putting them on squared paper to measure their perimeter: how come we get the same area but so much different boundaries in some figures and not in others…?

Grade 6, 4 hours work, 16 pupils involved

I actually had observed in the past that pupils often meet difficulties in imagining geometrical figures beyond the book-workbook-geometry lesson context. In some cases I happened to have to guide them to recognise figures they had already drawn in the Technical education class and they only had to reproduce for the geometry class.

The class in which the activity was carried out is mainly composed by pupils coming from the same primary school class, who had already worked with Tangram, as I came to know at the beginning of the lesson. I thought it would be better not to make pupils work in the same way they used in primary school, so we moved to the computer laboratory where we connected to a web site[28] which presents a game enabling pupils to play with Tangram's seven pieces to re-create either fancy or geometrical equally extended figures. Pieces can be rotated (by 45° each time) translated or, only in the case of parallelogram, overturned.

[28] http://www.math.it

The activity was fun for everybody and raised interesting remarks, such as for example:

“it is weird for geometrical figures to turn around”

“[parallelogram] fits in if I turn it upside down; it is as if it changed shape”.

In general it seemed to me that they were all engaged and we progressively commented that all the figures were obtained starting from the same modules by means of translations, rotations, flipping over without deformation. Also an autistic grade 8 pupil of mine took part in the activity and she amazingly could carry out correctly and quickly most of the game.

Grade 7, 5 hours, 15 pupils involved

Posed questions were understood by everybody. Also those who meet more difficulties in the class work participated autonomously and often found correct solutions.

In the first lesson pupils were invited to use the two equal isosceles triangles, putting the congruent sides aside to get the greatest possible number of different shapes […] I asked them to reflect on how to check whether figures were isoperimetric or not. The class thought they could use a ruler to measure the sides’ length, but when they found out that some dimensions were expressed in decimals, they decided to assume an arbitrary value as unit of measure, i.e. they assigned the smallest dimension the unitary value (we talked about it together) and by using Pythagoras’ theorem they found the other lengths.

I then asked whether the figures were equivalent. Only 10% of pupils answered correctly, so I needed to revise what I had done previously, suggesting they could work measuring the number of (paper) squares. The following week we used a square and a triangle, following the same method used in the previous class. This time the 85% answered correctly the question whether obtained figures were equivalent and isoperimetric.

After two days I asked them to form a rectangle, using all the Tangram pieces. After an initial critical moment they found 2 or 3 ways of getting it. I asked if the rectangle and square they had got from the pieces were equivalent and isoperimetric. In this case, they all answered correctly.

In general, regardless of the number of correct or wrong answers, I noticed that due to what they discovered in this activity pupils were led to reflect more before expressing the position. We got many solutions, although they were not extremely different. It is interesting that nobody thought about copying from their deskmate, as if the object under consideration were something personal. They certainly collaborated, but in a functional way to their solution needs. In particular a very good pupil (female) could not solve the problem [to find the area of one of the obtained figures] because she could not think using pieces. Some days later she confessed to me that when she solves a geometrical problem she only draws the figure to make me happy. Another girl, rather low-achieving, quickly solved the problem flipping a triangle (that she named figure 1) over the parallelogram (figure 2) writing down $A_2 = 2A_1$.

Grade 7, 4 hours for the initial phase + other 4 for tutoring, 14 pupils involved

[The initial phase was very similar to that carried out in grade 6 classes, also due to a certain general weakness of the class, which was involved in an activity of peer tutoring with pupils from a grade 3 primary school class (aged 8). The teacher thought she could use this tutoring to stimulate pupils to re-elaborate on their notions on a metacognitive level, in order to explain them to younger pupils.] The tutoring activity was proposed in two grade 3 classes and followed two distinct phases.

In the first phase, pupils, guided by older students, drew a 8 by 8 squares Tangram on squared paper with 1 cm side squares; the Tangram was then cut out and younger pupils invented and constructed different figures with the 7 pieces, giving each creation a “title”. At the end the children drew these figures on their workbooks.

The second phase envisaged an “enlargement” work, designed in collaboration with the Technical education teacher: “giant Tangrams” measuring 60x60 cm were constructed on 2.5 by 2.5 cm squared paper sheets. Tangrams were glued on cards and then cut out; each child was invited to recompose the figure already constructed and colour it as they fancied. The various pieces of each figure were fixed with adhesive tape, made stiff through bamboo sticks and then wore as masks.

At the end of the work there was a collective discussion in which younger pupils expressed their amazement in discovering that from initially identical Tangram they could make up such different figures. Older pupils tried to help the younger to understand why “some shapes look longer, even though they cannot have grown”, what changed from the beginning, when there was no difference between Tangrams. Among the sentences that most convinced young children we report the following, that seem to show an understanding of the work done and a non trivial capacity of verbal re-elaboration:

“Shapes are as big as earlier but the position of pieces changed”

“Parts without card changed, i.e. empty spaces” (of course behind this sentence hides the concept of equal extension, as it was clear to the boy who uttered it …).

COLLECTIVE DISCUSSION OF FEEDBACK

After trainees who experimented the activities in the classroom presented their reports, discussion focused on the motivational value of the activity (and everybody agreed on that), in particular on its potential to involve also pupils with scarce interest or capacity in mathematics. Much more interesting were pupils of different school level’s reactions: in the preliminary discussion some trainees had predicted that older students would have been less interested. This prediction was not confirmed, although it was noticed that younger pupils were actually more involved in the construction of fancy figures, whereas older pupils were soon ready to move to work on geometrical figures.

In this phase we also noticed how this work naturally stimulates the acquisition of techniques, methods and terminology linked to geometrical transformations. Hence the proposal was made to consider, in classes after grade 6, the activity as preparatory for a module of laboratory-like geometry, to be located after completing the polygon teaching unit and aiming to a reflection on equal extension and isometries (in particular symmetry, translation, rotation), as well as to reach competencies related to visualisation and recognition of geometrical figures in general.

PROPOSALS FOR FURTHER DEVELOPMENTS

At the end of the concluding discussion two further activities were proposed, one designed and partially already carried out by one of the trainees and the other presented by SSIS lecturers:

- <u>Tangram Web Quest</u>. If you digit the word Tangram on any Internet search engine you get the list of a great quantity of web pages, many of them suggesting teaching activities. A suitable activity for trainees might be to identify the most significant activities for learning at their pupils' school level; for pupils one might think about suggesting they connect to sites that contain a particular type of information or requests that enact further searches or readings.
- <u>Tri-dimensional puzzles</u>. Some pupils, as well as adults, have strong capacities for spatial view and graphical representation; some others meet difficulties on this ground. And it is well known that these capacities are not necessarily at the same level as other mathematical abilities. There are students with high competencies on the verbal plane and for whom it is easier to memorize a sentence like "a solid with 8 vertices" rather than visualize its image; vice versa, one can perfectly visualize a cube and needing to count vertices and sides every time…. This difference in cognitive styles makes it necessary to propose to all students activities involving spatial view and verbal description of solids, so that students can complement their competencies and also weaker students on the computational and algebraic planes, but strong in this other field can perform highly.

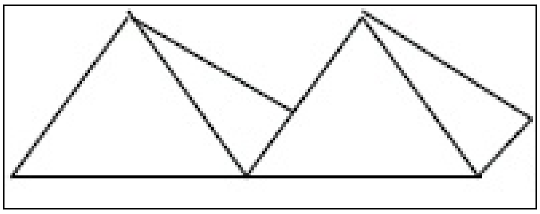
To exemplify the mechanism, the following questions were proposed to SSIS students:

> A. – "*Imagine a tetrahedron and write down how many faces, sides and vertices it has. Imagine you open up the tetrahedron so that you get its plane unfolding. What is its shape? Is there only one?*
>
> *A boy constructed a figure using squares and equilateral triangles, we do not know how many. We know that this figure has 5 faces, 5 vertices and 8 sides. What figure is it?*"

The question was not given with drawings, but it left everything to the involved SSIS students' spatial visualization.

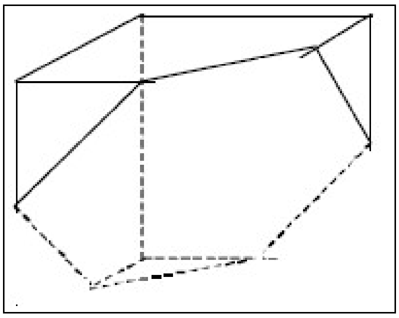
SSIS students felt the need to clarify that a tetrahedron is a pyramid with triangular basis and four faces ("like the silicate ion of quartz" one of the trainee suggested, drawing on her degree in Chemistry) in order to be able to solve the first part of the question relatively easily; further steps to solve also other parts followed, with the support of hands for "constructing" the object in the air.

B. – "*Imagine two different pyramids with squared basis, whose side faces are equilateral triangles. Put the two pyramids on a plane, getting them aside, so that they only share one* (*and only one*) *side of the basis. There is an empty space between these two solids. Would you be able to describe the solid that can fill in that emptiness, so that a convex solid can be obtained?*"



Picture 21. Two 3D pyramids

C. – "*Take 3 squares 10 by 10 cm, on one side of each of them cut out a right triangle with side 5. Consider also a regular hexagon with side 5* $\sqrt{2}$*. Combine now these 7 pieces to construct a solid as the one shown in Picture 22. Putting two of these solids aside what regular solid do we get?*"



Picture 22. Our solid figure with 7 faces

The answers to questions B. and C. (respectively, a tetrahedron and a cube) is not intuitive and this type of exercises exemplifies to trainees those difficulties in spatial visualization we referred to earlier; at the same time, both among trainees and in the classes where the activity was implemented, some manage to "see" the solution much earlier than others (sometimes surprisingly) and immediately become tutors and leaders for classmates.

D. – "*Take simple cubes* (*like wooden blocks*) *and try to construct a solid with a fixed number of these cubes. Represent then this solid from the different possible perspectives* (*frontally, from the right side, from the left side, from the top*) *using a given dotted grid. Conversely, given its representations, reconstruct the solid.*"

Of course in this type of activity, the main difficulty lies in having representations on different planes, some of which hidden to sight, and thus requiring a great effort for spatial representation. However this activity is also suitable for creating links to other disciplines like Technical education and Arts, beyond offering a good support to a rational description of what is actually achieved each time.

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Third piloting (at South Bohemian University, České Budějovice, CZ) and conclusion

by Jaroslava Brincková and Iveta Dzúriková

The general aim of the proposal Tangram piloted in Slovakia was to make the teacher trainees think of the importance that problem activities of measuring can bring into the mathematics formation of the pupils. We used the game of Tangram in seminars for the teacher trainees in their preparation for teaching static and metric geometry to pupils aged 11–14, that is, at lower secondary school. The main goal was the development of creative thinking and geometric imagination of pupils via using Tangram at school. We aimed at preparing a school activity in which we could deal with the concepts of perimeter and area in different contexts. We wanted to use Tangram to demonstrate isometric transformations in measuring perimeter and area, too.

We focused on the following partial aims:

- Didactic clarification of the sequence of steps in modelling the geometrical terms of perimeter and area of planar shapes*: perception – modelling – drawing in the plane – measuring – derivation of functional relations.*
- Describing van Hiele levels of geometrical thinking, in particular with focus on deduction of functional relations using geometric terms.
- Modelling world of numbers and shapes using the term *measure of abscissa*.
- Finding relations between perimeter and area of different shapes.

Partners in Florence (Italy), who co-piloted the project Tangram, provided us with the following feedback (their view of the project):

The proposal is related to geometrical properties of drawings, in particular to the measures of area and perimeter and to isometric transformations. It is planned as a laboratory activity, so that pupils have to use their perceptive, manual and logical skills, starting from concrete objects to achieve geometrical and graphical competences. Pupils, at the end of activity, are expected

- *to know basic names and notions about some common polygons*
- *to measure the length of a segment (directly or, if necessary, by means of Pythagoras Theorem)*
- *to realize the equivalence of plane figures which can be decomposed into the same parts*
- *to work on plane figures by means of isometric transformations and their compositions, realizing that new figures are congruent to the previous ones.*

Our partners in the Czech Republic cooperated with another teacher training institution (South Bohemian University, České Budějovice, teacher trainer Helena Binterová) in order to co-pilot the Tangram project. They provided us with the following variation of project aims:

Make future elementary school teachers familiar with the didactic means "Tangram" so that they are able to use it later in their teaching, in lessons of plane and metric geometry. The main aim was to define concepts, to develop creative thinking and geometric imagination and to make the student teachers aware of related didactical difficulties.

One of the compulsory assignments for the teacher trainees was to prove Pythagoras' theorem as well as sketch and justify the selected procedure.

The aims of all three participants of the project Tangram have been essentially identical. The pupils could develop their geometrical imagination via a didactical game and to strengthen their knowledge from isometric and metric geometry.

The teacher trainees prepared differentiated classes. They studied the problem of mapping in metric geometry via a priori analysis. They could see the theme of modelling in geometry from a new perspective via a posteriori analysis.